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Research paper

How do preservice mathematics teachers analyze and respond to student errors in solving probability problems using tree diagrams?

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ABSTRACT

This case study examined the knowledge of preservice mathematics teachers (PMTs) in analyzing and responding to student errors. Participants were 41 master's level PMTs who had completed foundational mathematics education courses. They evaluated hypothetical student work solving a compound probability problem using a tree diagram. Through fine-grained qualitative analysis of written responses, researchers found PMTs proficiently identified and interpreted errors, employing both "show-tell" and "give-ask" response strategies. Key weaknesses included over-emphasis on procedures, low active utilization of errors, and communication barriers. The study highlights implications for enhancing PMTs' error-handling skills in teacher education and improving probability teaching.

1. Introduction

Probability is a branch of mathematics that addresses topics such as random events and uncertainty (Korkmaz & Alkan, 2023), and its reasoning structure differs from that of other mathematical branches (Álvarez-Arroyo et al., 2024). These characteristics present a significant challenge for students, who frequently commit errors in learning this domain (Chernoff, 2012; He & Chen, 2025; Park & Lee, 2019). In particular, probability problems in compound experimental contexts may prove more challenging for students due to the construction of more complex sample spaces (Batanero et al., 2018; Chernoff & Zazkis, 2011; He & Chen, 2025; Landín & Salinas, 2018). For example, the problem of removing two balls from an opaque box containing two black and two white balls to determine the probability of a black-white pairing (a compound experimental context) is more challenging than the problem of removing one ball from an opaque box containing two black and two white balls to determine the probability of a black ball (a simple experimental context). To better solve such problems within complex experimental contexts, tree diagrams have been introduced as conceptual instruments for counting sample spaces (China Ministry of Education, 2022; Even & Kvatinsky, 2010). Nevertheless, existing research has demonstrated that students commit errors when using tree diagrams to solve more complex probability problems (Batanero et al., 2018).

To help students overcome errors, teachers must know how to diagnose student errors, analyze their underlying causes, and design appropriate interventions (Font et al., 2024). For preservice mathematics teachers (PMTs), diagnosing and analyzing student errors offers insight into their conceptual understanding (Font et al., 2024). Consequently, it is beneficial for PMTs to develop knowledge about identifying errors and developing corrective strategies (Brodie, 2014; Font et al., 2024; Korkmaz & Alkan, 2023). However, as Batanero and Álvarez-Arroyo (2024, p. 14) have highlighted in a recent retrospective review, "there are still very few papers centered on how teachers conceive their students' learning, predict their difficulties and strategies, and instructional practices to overcome these problems." Although several studies have investigated how PMTs address student errors, such as equiprobability bias, in learning probability (Park & Lee, 2019), to our knowledge, none have focused explicitly on how PMTs address student errors in solving more challenging probability problems.

This study aimed to examine how PMTs analyze and respond to student errors when solving probability problems using tree diagrams. This study contributes to the existing literature on how PMTs address student errors in learning probability, providing valuable insights into preparing PMTs for teaching probability.

2. Literature review and framework

2.1. Student errors in solving probability problems using tree diagrams

Large volumes of empirical research have identified probability as challenging for students, with various errors in their learning (e.g., Batanero & Chernoff, 2018; Park & Lee, 2019). When calculating the

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probability of an event mathematically, one constructs the sample space, that is, one finds the number of all equal possible outcomes of the experiment (m) and finds the number of the outcomes involved in the desired event (n), and then concludes a numerical value for the probability based on the formula n/m (Batanero & Díaz, 2007; Bryant & Nunes, 2012; He & Chen, 2025). However, students may make errors in the above process because they cannot construct sample spaces correctly. For example, the experiment of removing a ball from each of two boxes, both containing a black ball and a white ball, generates four equal possible outcomes, the collection of which is called the sample space, namely, {black-white, black-black, white-white, white-black}. Based on this sample space, the probability of the event "black-white pairing" is 2/4. Unfortunately, some students erroneously construct a sample space of {black-white, black-black, white-white} and thus believe that the probability of the event "black-white pairing" is 1/3 (He & Chen, 2025).

In response to such erroneous reasoning, the use of tree diagrams is considered an enlightening and visual tool that enables to prevent students from making errors in constructing sample spaces (Batanero et al., 2005; Böcherer-Linder et al., 2018; English, 2005; Landín & Salinas, 2018; Maher & Ahluwalia, 2014) and improve their attitudes and beliefs about probability (Williams & Nisbet, 2014). To illustrate, in addressing the problem above, one may decompose the experiment into two steps. Initially, it can be determined that there are two equally possible outcomes for the ball removed from one box, allowing for the drawing of two branches. The endpoint of each branch is referred to as a leaf node, which represents a single outcome within the sample space of the experiment of removing a ball from a single box. Subsequently, it is determined that there are two equally possible outcomes for the ball removed from the other box. Two new branches are then drawn at the nodes of each of these two branches, with the two leaf nodes at the end of these branches representing the two outcomes of the sample space for the experiment of removing a ball from the other box. It is thus hoped that students will be able to identify four possible outcomes for the experiment of removing two balls based on the entirety of the tree diagram, thereby avoiding the error of assuming that the probability of the event "black-white pairing" is 1/3.

The use of tree diagrams to represent and solve probability problems provides students with an enlightening visual tool that elucidates the structure of probability and facilitates an understanding of the mathematical calculation of probability, and thus promises them to avoid resorting to untenable intuitions when solving problems (Aspinwall & Shaw, 2000; Batanero & Álvarez-Arroyo, 2024; Böcherer-Linder et al., 2018; Landín & Salinas, 2018; Maher & Ahluwalia, 2014; Munter, 2014; Zahner & Corter, 2010). In contrast to an unorganized list of outcomes, using tree diagrams "helps students to easily represent the stages of a compound experiment" (Landín & Salinas, 2018, p. 258), thereby enabling students to visually and systematically count the sample space and to avoid omissions or duplications in enumerating the outcomes.

Nevertheless, they remain vulnerable to errors in using tree diagrams if the probability problem becomes more complex (Batanero et al., 2018). Students may suffer from omissions when drawing a tree diagram, thus forgetting to consider drawing branches and leaf nodes representing certain outcomes. To illustrate, in the context of removing a ball from a box containing two black balls and one white ball, students may represent the removal of the black ball and the white ball by drawing two branches, failing to recognize that the two branches are not equally probable.

Furthermore, students may encounter difficulties in interpreting the tree diagram. In the aforementioned context, despite the accurate representation of a tree diagram, students may not fully understand the significance of individual branches and leaf nodes, which can lead to errors when calculating the probability of compound events. In particular, when solving problems in which the probabilities of individual branches are unequal, for instance, when attempting to solve the problem "a shooter has a probability of 0.7 to hit the target and 0.3 to fail. What is the probability of hitting the target exactly three times out of five shots?" (Sánchez & Landín, 2014, p. 588), students commonly fail to recognize that the probabilities of the branches are not equal (Sánchez & Landín, 2014).

2.2. *Teachers' knowledge of student errors in learning probability*

Following the mathematical knowledge for teaching (MKT) framework proposed by Hills et al. (2008), proficient mathematics educators must possess both robust subject matter knowledge (SMK) and pedagogical content knowledge (PCK). It is widely acknowledged that "teachers cannot help children learn things they themselves do not understand." (Ball, 1991, p. 5). If teachers possess erroneous mathematical knowledge, they are likely to disseminate these misconceptions to their students during instruction, thereby hindering their learning (Copur-Gencturk, 2021; Hu et al., 2022).

Moreover, teachers' PCK also proves to be a significant factor influencing the teaching practice of mathematics (Baumert et al., 2010; Depaepe et al., 2013). Within the structure of PCK, there is a domain known as knowledge of content and students (KCS), which concerns teachers' knowledge of the process of student learning and errors encountered, playing an essential role in instructional practices aimed at addressing student errors (Greefrath et al., 2022; Hill et al., 2008; Pankow et al., 2018). The importance of teachers' knowledge about student errors is well documented. Several studies have highlighted the positive impact of teachers' knowledge of student misconceptions (KOSM) on facilitating students' shifts in misconceptions and enhancing conceptual understanding (Hill & Chin, 2018; Sadler et al., 2013).

In light of these perspectives, teachers must possess a robust understanding of probability and the requisite knowledge to address student errors in learning probability (Estrada et al., 2018; Park & Lee, 2019). It has been demonstrated that PMTs can identify student errors in solving simple probability problems. However, their ability to identify student errors diminishes when confronted with more complex or challenging problems (Chernoff & Zazkis, 2011; Park & Lee, 2019). Tree diagrams are commonly used to solve probability problems within complex and challenging experimental contexts. However, to our knowledge, scant research has examined the analyses and responses to student errors in probability problem-solving using tree diagrams, particularly in the context of PMT education.

2.3. *Analytical framework*

By integrating the perspectives of Peng and Luo (2009) and Son (2013), Hu et al. (2022) proposed a framework for examining teachers' analyses of and responses to student errors, as shown in Table 1.

According to this framework, teachers' analyses of student errors encompass three key facets: identifying, interpreting, and evaluating errors. The term "identify error" denotes detecting and explicitly locating the error. "Interpret error" signifies the elucidation of the

Table 1
Framework for examining teachers' analysis of and responses to student errors (Hu et al., 2022).

Aspect	Stages or facets	Descriptions or classifications
Analysis	Identify	Articulate student errors.
	Interpret	Analyze potential causes of the errors.
	Evaluate	Positive evaluation or negative evaluation.
Responses	Mathematical focus	Conceptual knowledge or procedural knowledge.
	Pedagogical actions	Specific strategies to address student errors.
	Address form	Show-tell or give-ask.
	Error use	Active use, medium use, or rare use.
	Communication barrier	Over-generalization, Plato-and-the-slave-boy approach, return to the basics; Specific to the student error.

underlying causes of the error. Finally, “evaluate error” pertains to teachers’ attitudes towards the error.

Teachers’ responses to students’ errors comprise five facets.

a) Mathematical focus denotes the specific mathematical knowledge the teacher prioritizes when addressing students’ errors. This encompasses both procedural and conceptual knowledge, where the former includes formulas, procedures, and other technical details, while the latter comprises definitions, connotations, and other fundamental concepts.
b) Pedagogical actions refer to the specific teaching strategies teachers adopt to correct student errors.
c) Form of address reflects whether teachers’ corrective teaching is student-centered or teacher-centered. Categories include “give-ask” and “show-tell”. “Give-ask” focuses on student reflection, interaction, and discussion, while “show-tell” emphasizes teachers’ explanations of errors and demonstrations of correct solutions (Runnalls & Hong, 2020).
d) Degree of student error use, which reflects how teachers view errors as valuable resources when dealing with student errors, includes three categories: active use, medium use, and rare use. Active use implies that teachers are concerned with errors as the focus and reference of corrective teaching. In contrast, medium use implies that teachers only view errors as stepping stones for corrective teaching (Hu et al., 2022).
e) A communication barrier refers to the obstacles encountered between teachers and students when communicating about student errors. These obstacles can be classified into three categories: over-generalization, the Plato-and-the-slave-boy approach, and return to the basics. Over-generalization refers to teachers’ proposals of a program that is too general and lacks specific activities or tasks. The Plato-and-the-slave-boy approach is characterized by the belief of teachers that students have temporarily forgotten certain facts or procedures that have led to the error. This approach entails the teachers’ belief that the student only needs to be helped to recall these facts and procedures. “Return to the basics” is characterized by a focus on leading students to review basic concepts or principles while neglecting to address the problem at hand. Furthermore, to account for responses that effectively address the student’s error without exhibiting the defined communication barriers, a category labeled “specific to student error” was included in our coding scheme, following the methodological approach of Hu et al. (2022).

2.4. The present study

Competent mathematics teachers must be able to analyze students’ problem-solving thoughts and reasoning processes and provide effective interventions accordingly (Bas-Ader et al., 2024; Scheiner & Montes, 2024). Without well-trained teachers, the teaching of probability in schools would struggle to improve (Huerta, 2018). However, “research in teacher education related to probabilistic thinking and reasoning has been identified as scarce” (Ingram, 2024, p. 1). The present study examined PMTs’ knowledge about addressing student errors in solving probability problems through their written responses to a hypothetical student work. We proposed two research questions.

(1) How do PMTs analyze student errors in solving probability problems using tree diagrams?
(2) How do PMTs respond to student errors in solving probability problems using tree diagrams?

3. Methods

3.1. Participants

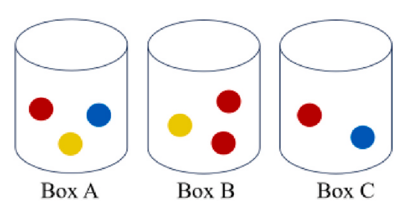
A total of 41 master’s level students in mathematics education, comprising 29 females and 12 males, participated in this study. Following the program of the host university, the training cycle for PMTs at the master’s level is two years. During the first year, they engage in a series of courses related to mathematics education, including Design and Practice of Mathematics Teaching, Analysis of Mathematics Curriculum Standards and Textbooks, Mathematics Education Psychology, and Research Methodology in Mathematics Education. Additionally, they undertake a one-semester internship at a middle or high school and complete a thesis in their second year. The participants involved in this study had completed all the requisite first-year courses in the program but had not yet commenced their internships. Specifically, prior to participating in this study, they had systematically studied Probability Theory as part of their undergraduate program and completed a one-year internship at a middle or high school. During their master’s program, they have learned about probability content for elementary, middle and high school levels through the Analysis of Mathematics Curriculum Standards and Textbooks course; theoretical knowledge related to the psychological aspects of mathematics learning through the Mathematics Education Psychology course; and theoretical and practical methodologies for developing lesson plans through the Design and Practice of Mathematics Teaching course.

3.2. Materials

We presented participants with the task of solving a probability problem using a tree diagram and a hypothetical student work. We asked them to analyze the errors in the student’s solution and provide feasible pedagogical strategies to address these errors. This context for solving probability problems using tree diagrams is familiar in middle school probability classes (China Ministry of Education, 2022). The task and hypothetical student work are as follows.

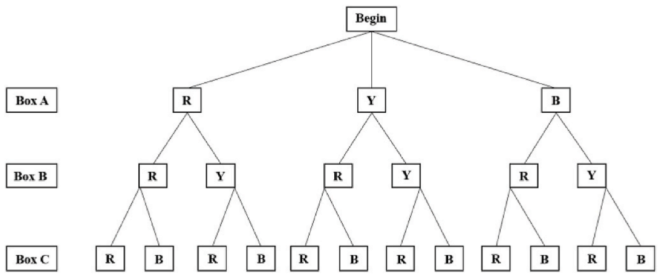
Shu is working on the following problem:

There are three opaque boxes: A, B, and C. Box A contains one red ball, one yellow ball, and one blue ball; box B contains two red balls and one yellow ball, and box C contains one red ball and one blue ball, all of which are identical except for their color. With your eyes closed, remove one ball from boxes A, B, and C and find the probability of the event “the three balls removed contain at least one red ball” using the tree diagram method.



Shu’s reasoning is as follows:

I solved the problem in two steps. I drew the tree diagram in the first step, as shown below. In the second step, I found that there are eight possible combinations of the three balls removed: {Red, Red, Red; Red, Red, Blue; Red, Red, Yellow; Red, Yellow, Blue; Red, Yellow, Yellow; Red, Blue, Blue; Yellow, Blue, Blue; Blue, Yellow, Yellow}. The event “the three balls removed contain at least one red ball” contains six combinations, as underlined. Therefore, the probability of this event occurring is 6/8.



Participants were asked to answer three questions.

(1) What do you think the answer to this question is?
(2) Do you think Shu's reasoning is correct? If not, please describe the error(s) reflected in his reasoning and explain why.
(3) What teaching strategies would you take to help him correct his error(s)?[1]

Shu's errors are reflected in two aspects. The first error (Error 1) is reflected in the first step, where the tree diagram he drew is incorrect. Given that there are two red balls in box B, the tree diagram should represent the probability of the red ball being removed as 2/3. This is typically achieved in two different forms. One approach is to draw three branches to illustrate that the red ball (which might be labeled Red-B1), the other red ball (which might be labeled Red-B2), and the yellow ball have an equal chance of being removed. Alternatively, the branch of the red ball can be explicitly labeled as having a probability of 2/3, allowing the product law of probability to be applied. The former approach is suitable for Chinese middle school students who have not been formally introduced to the concept of product law of probability.

The second error (Error 2) is reflected in the fact that, even if we assume that his first step was correct, he made an error in enumerating the possible combinations of the three balls removed according to the tree diagram. In other words, he constructed an incorrect sample space. According to the tree diagram he drew, the sample space should comprise 12 outcomes, each with an equal probability, rather than combining the four equal outcomes of "Red-A, Red-B1, Blue-C″, "Red-A, Red-B2, Blue-C″, "Blue-A, Red-B1, Red-C″, and "Blue-A, Red-B2, Red-C″ into a single outcome designated as "Red, Red, Blue." In a correct tree diagram, there are 18 possible combinations of the three balls removed. Thus, the sample space includes 18 outcomes of {Red-A, Red-B1, Red-C; Red-A, Red-B1, Blue-C; Red-A, Red-B2, Red-C; Red-A, Red-B2, Blue-C; Red-A, Yellow-B, Red-C; Red-A, Yellow-B, Blue-C; Yellow-A, Red-B1, Red-C; Yellow-A, Red-B1, Blue-C; Yellow-A, Red-B2, Red-C; Yellow-A, Red-B2, Blue-C; Yellow-A, Yellow-B, Red-C; Yellow-A, Yellow-B, Blue-C; Blue-A, Red-B1, Red-C; Blue-A, Red-B1, Blue-C; Blue-A, Red-B2, Red-C; Blue-A, Red-B2, Blue-C; Blue-A, Yellow-B, Red-C; Blue-A, Yellow-B, Blue-C}, thus the probability of the desired event is 16/18.

[1] It is noteworthy that the design of the third question implicitly suggests that error(s) exist(s) in the student's reasoning presented in the second question. We acknowledge that this could potentially influence participants' responses to the second question, as it may prime them to look for an error. However, this design was intentional and aligned with the study's objective, which focuses not on the detection of errors per se, but on the analysis and response to errors once they are identified. In authentic teaching scenarios, teachers are typically already aware that a student's answer is incorrect before devising an instructional response. Thus, the task sequence mirrors a realistic pedagogical situation where the teacher's goal is to understand and address a known student difficulty. The primary focus of our analysis was therefore on the depth and quality of the PMTs' error interpretation and their proposed teaching strategies, rather than on the simple act of error detection.

3.3. Data collection

The researcher informed the participants that the purpose of this investigation was to examine preservice teachers' pedagogical knowledge, and all participants provided written consent to participate in this investigation. Data were collected at the end of the semester when the participants had completed their first year of the training program. The participants were invited to complete a coursework assessment to evaluate the PMTs' PCK during a regular class lasting 60 min. Accordingly, the data analyzed in this study are the written responses provided by the participants. To collect as much data as possible, the researchers provided participants with sufficient answer sheets and encouraged them to use written words, illustrations, or other forms to express their thoughts fully. According to post-analysis, the average length of the participants' written responses exceeded 600 words. Pseudonyms were used when quoting participants' written responses.

3.4. Coding

Following the established analytical framework, four researchers coded the participants' written responses through an iterative review process. In this process, the four coders initially coded independently, revisiting the definitions of the coding categories whenever discrepancies emerged and discussing these discrepancies in depth until a consensus was reached (Syed & Nelson, 2015). In this study, the four coders examined the analytical framework in detail and underwent training on the coding criteria before commencing their work. They then confirmed that their understanding of the framework and criteria was consistent. During the coding process, approximately 10 % of their initial independent coding revealed discrepancies. However, after discussions and iterative reviews, these discrepancies were resolved, and a consensus was finally reached.

The first question was designed to examine whether the PMTs could solve the problem. It was coded as 1 for a correct response by the participant and 0 for an incorrect response.

The second question was designed to examine how PMTs analyze student errors, categorized into three distinct facets. The identification of errors was readily apparent from the written responses. The interpretation of errors was derived from an iterative qualitative analysis of the written responses related to specific errors.

The coding of PMTs' interpretations for Error 1 included three categories: a) The interpretations provided were overgeneralized (Interpretation 1); b) It is highlighted that Shu did not consider the equal probability of the three balls in box B being removed, and incorrectly assumed the three possible outcomes, namely, {Red-A, Red-B, Yellow}, to be {Red, Yellow}, leading to a flawed tree diagram (Interpretation 2); and c) It is acceptable that Shu drew the tree diagram with the branches of box B in two, but he did not explicitly label the probability of the red ball being removed as 2/3, resulting in an incorrect answer (Interpretation 3).

The coding of the PMTs' interpretations for Error 2 included three categories: a) The interpretations provided were overgeneralized (Interpretation 1); b) Proposing the correct solution to be used as an interpretation (Interpretation 4); and c) Suggesting that Shu did not take into account the equiprobability of the individual outcomes when constructing the sample space, and incorrectly treated some different outcomes as a single one (Interpretation 5). Concerning PMTs' evaluation of student errors, a similar approach was employed to that used by Hu et al. (2022), with the data coded into three categories: "positive," "negative," and "half-half."

The third question was devised to examine how PMTs respond to student errors, coded from five distinct facets. Following previous research (Runnalls & Hong, 2020), the mathematical focus was classified as "conceptual knowledge," "procedural knowledge," or "both." Specifically, if the relevant concepts pertinent to the problem were focused upon, including probability, sample space, and so forth, then it was coded as a focus on conceptual knowledge. Conversely, if the procedures, steps, and formulas necessary to solve the problem were the focus, then it was coded as a focus on procedural knowledge. Finally, if both conceptual and procedural knowledge were focused upon, it was double-coded as "both." The pedagogical actions were associated with specific errors. Based on an iterative analysis of the written responses, five activities were identified: a) The teacher introduces students to relevant knowledge such as probability and the sample spaces but forgets to solve the problem at hand, which can be summarized as "reviewing foundational knowledge" (Action 1); b) The teacher presents or clarifies the solution to the problem in certain manners and asks students to practice similar problems, which can be summarized as "clarifying the solutions" (Action 2); c) The teacher traces students' thoughts through a chain of questions, analyses the causes of errors and guides students to correct them, which can be summarized as "instruction through question chains" (Action 3); d) The teacher encourages students to engage in reflection and discussion with their peers to identify and correct errors, which can be summarized as "promoting reflection and discussion" (Action 4); e) The teacher provides students with the opportunity to engage in experimentation and to gain insight into probability through the collection and analysis of data, which can be summarized as "encouraging students to experiment" (Action 5). Both Action 1 and Action 2 fall within the pedagogical strategies of direct teaching or corrective feedback. The form of address was coded into three categories: "show-tell," "give-ask," or "both." Communication barriers were categorized into four types: over-generalization, the Plato-and-the-slave-boy approach, returning to basics, and specific to student error.

4. Results

4.1. PMTs' analysis of student errors

We first report in passing that all PMTs successfully solved this problem, demonstrating their robust SMK. Table 2 provides a detailed analysis of PMTs for the two errors, including their identification, interpretation, and evaluation.

Identify errors. Forty participants (98 %) identified Error 1, whereas only 25 (61 %) identified Error 2. Further analysis demonstrated that all PMTs identified at least one error, with 17 participants (41 %) identifying one error and 24 (59 %) identifying two errors. It is noteworthy that PMTs appear less effective at identifying error 2, which may be because some participants did not explicitly state the error despite being aware of it. This will be explained later.

Table 2
The participants' analysis of the two errors.

Facet	Classification		N (%) Error 1	Error 2
Identify	Identified		40 (98 %)	25 (61 %)
	No errors were identified		0 (0 %)	
	One error was identified		17 (41 %)	
	Two errors were identified		24 (59 %)	
Interpret	Interpretation types	Interpretation 1	2 (5 %)	3 (7 %)
		Interpretation 2	35 (85 %)	/
		Interpretation 3	4 (10 %)	/
		Interpretation 4	/	13 (32 %)
		Interpretation 5	/	22 (54 %)
	Number of interpretations	No interpretation	0 (0 %)	
		One interpretation	3 (7 %)	
		Two interpretations	38 (93 %)	
Evaluate	None		39 (95 %)	
	Negative		0 (0 %)	
	Half-half		2 (5 %)	
	Positive		0 (0 %)	

Interpret errors. Concerning Error 1, 35 participants (85 %) interpreted it as Interpretation 2. This interpretation assumes that Shu did not consider the equal probability of the three balls in box B being removed, thus leading to a flawed tree diagram. Four participants (10 %) interpreted it as Interpretation 3. This interpretation assumes that it is acceptable for Shu to draw the tree diagram with the branches of box B in two, but that he did not explicitly label the probability of the red ball being removed as 2/3. Consequently, an incorrect answer resulted. Two participants (5 %) provided overgeneralized interpretations (Interpretation 1). As can be observed, most participants could provide concrete and sensible interpretations for Error 1, which can be classified as either Interpretation 2 or Interpretation 3. In the case of Error 2, 22 participants (54 %) interpreted it as Interpretation 5, suggesting that Shu had not taken into account the equiprobability of the individual outcomes when constructing the sample space and had incorrectly treated some different outcomes as a single one. 13 participants (32 %) interpreted it as Interpretation 4, proposing the correct solution as an interpretation. The remaining 3 participants (7 %) provided overgeneralized explanations (Interpretation 1).

Notably, the written responses yielded explicit statements from most participants regarding the interpretation of Error 2. This suggests that these participants identified the error but did not explicitly indicate it in the response script. The two errors occurred in sequence during the problem-solving process, resulting in a chain reaction whereby the appearance of Error 1 affected the subsequent problem-solving procedure. Some participants may have overlooked the potential errors that emerged after Error 1 when reporting the errors they identified. To illustrate, Jingwen interpreted that, "Let us assume that the tree diagram was correctly drawn, but that such outcomes as 'Red, Yellow, Red' and 'Yellow, Red, Red' were treated as one. This remains an unsatisfactory solution." Subsequently, she elucidated, "He had already committed an error (Error 1) in drawing the tree diagram. To be precise, he was unaware that there were three equally probable outcomes of the ball being removed from Box B. Had he not made these errors, a total of 18 possible outcomes would have been enumerated, 16 of which consist of the desired event. Therefore, the correct answer is 16/18."

Furthermore, 38 participants (93 %) provided two interpretations, while only three (7 %) provided one. This again demonstrates that some participants, while identifying Error 2, did not explicitly indicate it.

Evaluate errors. In the final step of examining the PMTs' analysis of student errors, we attempted to capture opinions or information regarding the evaluation aspects of the errors from the participants' response scripts. Unfortunately, 39 participants (95 %), the vast majority, did not explicitly provide their evaluations of the errors.

4.2. PMTs' responses to student errors

Table 3 presents PMTs' responses to errors.

Mathematical focus. First, 23 participants (56 %) preferred to correct errors by reinforcing students' procedural knowledge. To illustrate, Jing proposed that "the first step should be to label the balls. Thus, the balls in box A could be labeled Red-A, Yellow-A, and Blue-A; the balls in box B could be labeled Red-B1, Red-B2, and Yellow-B; and the balls in box C could be labeled Red-C and Blue-C. Subsequently, the students were directed to redraw the tree diagram, which would reveal the inaccuracy of their previous approach." It appears that her primary concern was to instruct the students on how to follow the specified procedure to solve the problem, rather than to guide them in recognizing the necessity for labeling and the underlying concepts involved.

Second, 17 participants (42 %) were concerned with reinforcing students' conceptual and procedural knowledge to correct their errors in problem-solving. For instance, Qingwen proposed that "educators should emphasize to students the importance of ensuring that each outcome in the sample space is equiprobable when determining the

Table 3
The participants' analysis of the two errors.

Facet	Classification		N (%)
Mathematical focus	Procedural		23 (56 %)
	Conceptual		1 (2 %)
	Both		17 (42 %)
Pedagogical actions[a]	Action types	Action 1	18 (20 %)
		Action 2	33 (37 %)
		Action 3	8 (9 %)
		Action 4	11 (12 %)
		Action 5	19 (21 %)
	Number of actions	One action	41 (100 %)
		Two actions	32 (78 %)
		Three actions	11 (27 %)
		Four actions	5 (12 %)
Address form	Show-tell		15 (37 %)
	Give-ask		0 (0 %)
	Both		26 (63 %)
Errors use	Rare use		1 (2 %)
	Medium use		22 (54 %)
	Active use		18 (44 %)
Communication barrier	Over-generalization		12 (30 %)
	Plato-and-the-slave-boy		0 (0 %)
	Return to the basics		3 (7 %)
	Specific to student error		26 (63 %)

[a] Since some participants proposed multiple actions, 89 valid records were captured. It should be noted that when counting the percentage of actions from 1 to 5, the denominator is no longer the number of participants but the total number of records.

sample space. To address errors effectively, it is essential to undertake two key steps. Initially, it is vital to facilitate reflection on whether the ball removed from box B has only two equally probable outcomes, as illustrated in the tree diagram created by Shu. This step warrants particular emphasis, as it is only when students can correctly enumerate the sample space, confirming that all listed outcomes are equally probable, that they can accurately calculate the probability. Subsequently, the students should be guided to label the balls and draw a tree diagram that represents the experiment in an organized manner."

Third, a mere one participant (2 %) identified the reinforcement of students' conceptual knowledge as a means of correcting errors.

Pedagogical actions. First, most PMTs preferred to respond to student errors with pedagogical actions, such as direct teaching or corrective feedback. The most frequently reported action was Action 2, "clarifying the solutions," which occurred 33 times (37 %) out of 89 valid records. In addition, Action 1, "reviewing foundational knowledge," was observed on 18 occasions (20 %).

Second, Action 5, "encouraging students to experiment," occurred 19 times (21 %). For example, Yong suggested, "I would simulate the context by giving students the box and the balls and then asking them to conduct multiple experiments to gain experience. In particular, I would allow them to experience that the red ball is removed from box B more frequently than the yellow ball, thus noticing that the tree diagram Shu drew is wrong. In addition, I would consider resorting to software to develop a simulated application of this context to help the students notice Shu's errors through computer-simulated experiments."

Third, Action 4, "promoting reflection and discussion," occurred 11 times (12 %). For example, Feiyang suggested, "In response to the first error, I would ask him, 'If there are ten red balls and one yellow ball in box B, and a ball is randomly removed from the box, what is the probability that the removed ball is a red one?' I think the disparity in the number of red and yellow balls could make the students think differently. For the second error, I'd suggest playing a game of tossing two coins and then asking the students what the probability of a heads-tails pair is. If we follow Shu's logic, the answer would be 1/3. I'd then steer the discussion towards whether there's a difference between the head-tail and tail-head outcomes. Finally, I'd like to remind them that 'Red, Red, Yellow' and 'Red, Yellow, Red' are not the same outcome."

Fourth, Action 3, "instruction through question chains", was mentioned the least, at eight times (9 %). Fifth, in terms of the number of activities proposed, all 41 participants (100 %) proposed at least one action, 32 participants (78 %) proposed two actions, 11 participants (27 %) proposed three actions, and 5 participants (12 %) proposed four actions.

Address form. First, 26 (63 %), the majority, of the participants adopted both the "show-tell" and "give-ask" forms. Second, 15 participants (37 %) adopted the "give-ask" as the sole form. Third, none of the participants adopted the "show-tell" as the sole form.

Errors use. First, 18 participants (44 %) viewed student errors positively, treated them as the focus of instructional interventions, and taught against them (active use). Second, 22 participants (54 %) viewed student errors as a stepping stone (medium use). Third, only one participant (2 %) completely disregards student errors (rare use).

Communication barrier. First, 12 participants (30 %) revealed the communication barrier of over-generalization when responding to student errors. Second, three participants (7 %) revealed the communication barrier of returning to the basics when responding to student errors. Third, the participants' response scripts did not identify the communication barrier associated with the Plato-and-the-slave-boy approach. Fourth, 26 (63 %), the majority of the participants did not encounter a communication barrier when responding to student errors.

5. Conclusion, discussion, and implications

We begin with answers to the research questions, followed by a discussion of the findings, implications, and contributions of this study.

5.1. How do PMTs analyze student errors in solving probability problems using tree diagrams?

The findings of this study demonstrated that, although some participants did not explicitly articulate Error 2, they could identify the error through the lens of the subsequent interpretation of the error. Therefore, PMTs performed well overall in identifying student errors. However, we must acknowledge that there may be a discrepancy between the identification of student errors by PMTs and their explicit articulation of these errors. The phenomenon may be attributed to the chain effect of errors, whereby initial errors lead to a subsequent series of errors. Some participants may have focused solely on the initial error, thereby overlooking subsequent errors when representing student errors. This finding reflects their lack of awareness of categorizing and recording errors. It is therefore recommended that examples of analyzing students' errors be included in the PMTs' training program, with opportunities provided for them to practice analyzing student errors in various contexts. Particular guidance should be provided on the accuracy and completeness of error identification, as well as on explicitly identifying, articulating, and communicating students' errors.

The findings of this study indicated that most participants could accurately interpret the underlying causes of Error 1. The participants proposed that the student failed to clarify that the chances of the red and yellow balls being removed from box B were not equally probable. Consequently, they recommended that either a new branch be added or that the probabilities of the two branches be labeled when drawing the tree diagram (Interpretation 2 and Interpretation 3). Over half of the participants could provide concrete interpretations of the causes of Error 2 (Interpretation 5). They argued that the students did not understand that all the outcomes in the sample space needed to be equally probable. The capacity of the PMTs to accurately interpret student errors may be associated with their robust SMK. This might be evidenced by the fact that they were all able to solve the problem correctly. Furthermore, the results of this study indicated that most participants provided more than one interpretation. The causes of student errors in problem-solving can be multifaceted, each suggesting different pedagogical implications (Shaughnessy et al., 2021). Once again, the capacity of PMTs to offer multiple interpretations of student errors reflected their proficiency in analyzing errors from a student perspective.

The findings of this study indicated that the overwhelming majority of PMTs did not explicitly evaluate student errors. This may be because we did not explicitly request that participants express their attitudes toward student errors. We speculate that if the participants had been explicitly asked to express their attitudes toward student errors, they might have expressed various opinions, including some positive ones. In other words, we may have underestimated the proportion of PMTs with positive attitudes towards student errors. However, this finding also precisely reflected the lack of awareness of proactively evaluating student errors. This finding is consistent with existing research indicating that some teachers may perceive errors as an indication of failure, leading to a reluctance to confront student errors or a preference for focusing on correcting them rather than utilizing them as valuable learning opportunities (Shaughnessy et al., 2021). This phenomenon of PMTs' lack of awareness in evaluating student errors underscores the necessity for enhancing their perception of the significance of student errors and their skills in offering constructive feedback. Teacher training programs must, therefore, place greater emphasis on these aspects. Moreover, teacher educators must encourage PMTs to engage in reflective practice concerning student errors, in addition to identifying and interpreting them. This should be done to promote a positive attitude towards such errors. A positive evaluation of student errors has been demonstrated to promote learning and foster self-confidence. When teachers adopt a positive and encouraging approach to evaluating student errors, students perceive the teacher's emotional support, which in turn enhances their motivation, engagement, and persistence (Heinze et al., 2012; Käfer et al., 2019; Soncini et al., 2021). Research has demonstrated that teachers' positive attitudes towards student errors, such as respect, acceptance, and tolerance, can reduce students' anxiety or fear of making errors. This, in turn, can enhance risk-taking attitudes and persistence in problem-solving (Tulis, 2013). Teachers' negative and problematic attitudes toward student errors can hinder the creation of a supportive learning environment for their students (Leighton et al., 2022). In light of this, it is hypothesized that teachers' friendly, inclusive, and positive view of student errors can facilitate the formation of learning environments that encourage creativity and exploration and facilitate meaningful reflection on the process of making errors. It is, therefore, essential to provide PMTs guidance to promote the development of an optimistic, inclusive, and constructive attitude towards errors (Hu et al., 2022).

5.2. How do PMTs respond to student errors in solving probability problems using tree diagrams?

This study's findings indicated that most participants preferred to address student errors by reinforcing procedural knowledge, suggesting a tendency to prioritize the acquisition of correct procedures over the development of conceptual understanding. This finding is consistent with existing research indicating that teachers prioritize enhancing procedural knowledge when addressing student errors (Runnalls & Hong, 2020; Son, 2013; Stohl, 2005). While procedural knowledge is essential for solving specific problems, conceptual knowledge provides students with a foundation for understanding the principles and logic behind the problems. The resolution of probability problems cannot be accomplished by merely applying algorithms and formulas; rather, it necessitates establishing a coherent structure of probabilistic thinking (Erbas & Ocal, 2022). Prior research has shown that a dual pedagogical approach, which emphasizes both procedural and conceptual knowledge, enhances students' capacity to comprehend mathematical concepts and refine their problem-solving skills (Runnalls & Hong, 2020). It is encouraging to observe that some participants demonstrated an awareness of the significance of conceptual knowledge and adopted a more comprehensive and balanced approach to correcting the errors. This approach reflected a more holistic perception of the nature of student errors and a recognition of the importance of fostering a deeper conceptual understanding of mathematical concepts among students. This dual focus is crucial because it involves not only solving the probability problem with the help of a tree diagram but also applying probabilistic thinking and the underlying principles that underpin the construction of this tree diagram.

The findings of this study indicated that "reviewing foundational knowledge" and "clarifying the solutions" dominated all the proposed teaching actions, suggesting that PMTs favor presenting students with clear explanations and paradigms when correcting errors. This phenomenon may be related to traditional practices in Chinese mathematics education that emphasize basic knowledge, problem-solving solutions, and practice (Sun, 2011; Tang et al., 2013). Alternatively, it may be linked to the pedagogical tradition in Chinese mathematics education that emphasizes using exemplary lessons (essentially a standard, excellent example) to enhance teaching practice (Huang et al., 2013; Niu et al., 2017). Viewing these two teaching activities as conservative or backward would be inappropriate. Indeed, recent studies have indicated that the absence of substantial analysis of the concepts or principles behind errors in correcting students' errors may limit the opportunities for students to learn from their errors (Alvidrez et al., 2024).

However, this study observed that PMTs placed less emphasis on "instruction through question chains" (Action 3) and "promoting reflection and discussion" (Action 4), suggesting that PMTs may have relied too heavily on reinforcing the basics or providing exemplary patterns (so-called standard reasoning) while neglecting the positive effects of other, more inquiry-based, instructional approaches for correcting students' errors. Only when teachers design activities that provide opportunities for students to engage in thinking and reflection can they become active learners (Ahuja, 2018). It is noteworthy that "encouraging students to experiment" (Action 5) is not a frequently mentioned approach; however, it is encouraging as it has been demonstrated to have a positive effect on the visual perception of probability in students (Nilsson et al., 2018; Paparistodemou & Meletiou-Mavrotheris, 2008; Park & Lee, 2019). China's most recent mathematics curriculum standards explicitly recommend that teachers integrate and coordinate multiple pedagogical approaches, including direct and non-direct teaching, to promote students' mathematics learning (China Ministry of Education, 2022). The study revealed that some participants proposed various teaching activities, indicating that PMTs are flexible in addressing student errors and are conscious of integrating diverse pedagogical approaches to meet students' needs and address multiple errors. This finding is encouraging as it suggests that PMTs are aware of the need to develop diverse pedagogical strategies required for effective mathematics teaching. However, the relatively limited number of participants who suggested three to four activities indicates that there is scope for further expansion of their repertoire of pedagogical strategies. Therefore, training programs should provide PMTs with practical opportunities to apply these approaches in their teaching, thereby enabling them to gain exposure to and utilize a more diverse range of student-centered and inquiry-based teaching methods.

The findings of this study revealed that the combination of "show-tell" and "give-ask" was the most popular among PMTs in terms of address form. This reflects their dual emphasis on lecturing and asking questions, as well as their awareness of the importance of interaction between the teacher and students in corrective teaching. Some participants employed the unique address form of "give-ask," however, it is noteworthy that none of the participants utilized the unique address form of "show-tell," which may suggest that they recognize the limitations of the "show-tell" approach and the necessity to incorporate questioning and interaction into corrective teaching. The traditional lecturing approach may effectively correct students' errors immediately, but it has been demonstrated to have limitations in developing students' capacity for deep understanding, reflection, and critical thinking. Prior research has shown that traditional lecturing has a constrained impact on correcting students' errors. Conversely, indirect teaching approaches that prioritize student-centeredness and facilitate student interaction have been identified as more effective in promoting students' learning (Madu & Orji, 2015; Shaughnessy et al., 2021; Taslidere & Yıldırım, 2023).

This study's findings indicated that most participants did not ignore students' errors, which is encouraging and reflects PMTs' attention to the resources of errors. Errors should not be viewed negatively but rather as an essential resource for establishing students' mathematical comprehension (Ahuja, 2018; Alvidrez et al., 2024; Font et al., 2024; Hu et al., 2022; Shaughnessy et al., 2021; Stockero et al., 2020; Tan Sisman & Aksu, 2016), and can serve as guidelines for teachers to adapt instruction (Tulis, 2013). It was observed that some participants regarded student errors as the focus of instructional interventions (active use), indicating a readiness to view errors as a potential avenue for student learning and a foundation for corrective teaching. However, while not ignoring student errors, most participants only used them as a stepping stone (moderate use), indicating that PMTs may not fully recognize the potential of errors to facilitate conceptual comprehension. This finding is consistent with previous research, which stated that teachers do not favor using errors to promote student learning in many mathematics classrooms worldwide. Indeed, such errors are often viewed as a negative, embarrassing, or shameful aspect of the learning process (Alvidrez et al., 2024; Bray, 2011; Tulis, 2013). Therefore, teacher educators must facilitate a deeper comprehension of the effective utilization of errors among PMTs.

This study's findings indicated that most participants did not exhibit significant communication barriers in their responses to student errors. Another encouraging finding is that no such communication barrier as the Plato-and-the-slave-boy approach was captured from the participants' written responses. This suggests that PMTs can consciously pursue meaningful approaches to mathematical learning. However, it was observed that a proportion of participants exhibited a communication barrier of over-generalization, as evidenced by the provision of vague or overly generalized interventions, which lacked targeted specificity. This indicates the need for PMTs to enhance their capacity to furnish more detailed and tailored instructional responses to students, thus enabling the effective navigation of the nuances of student errors (Ahuja, 2018). Furthermore, a few participants demonstrated the communication barrier of "return to the basics," indicating an excessive focus on foundational knowledge without addressing the balance between reviewing foundational knowledge and solving the problem. Focusing solely on the basics may result in students acquiring knowledge that lacks practical significance and, therefore, lacks robustness.

To summarize, this study has three main contributions. First, as recent research has proposed, there is a paucity of studies on teachers' knowledge of students' probabilistic thinking, reasoning, and learning (Batanero & Álvarez-Arroyo, 2024; Ingram, 2024). The present study concentrated on the analysis and response of PMTs to the errors that occur when students utilize tree diagrams to solve probability problems in compound experiment contexts, specifically those with more complex constraints. The findings of this study contribute to the existing body of knowledge in this field, providing insights into the knowledge of PMTs regarding learning and teaching probability. Second, this study examined PMTs' knowledge of analyzing and responding to student errors in detail through a comprehensive framework that considers multiple facets simultaneously. Thus, the findings advance the understanding of teachers' PCK regarding this specific topic and provide insight into developing PMS's PCK. These insights provide a foundation for future research to further explore PMTs' PCK. Third, this study informs the design and improvement of training programs for PMTs, such as incorporating error correction activities to enhance their knowledge of student learning in mathematics.

6. Limitations and further research

This study has two limitations. The first is that the sample size was small; thus, when interpreting the results, it cannot be overgeneralized. The second is that the study relies solely on participants' written responses, which may not fully reflect their reasoning and thinking. For example, the present study has not captured enough information concerning the participants' evaluation of student errors. If they had been explicitly requested to express their attitudes, there might have been richer data to describe the error evaluation facet better.

Furthermore, it is possible that additional information might have been captured had further interviews been conducted with them. Future studies may consider expanding the sample size and adopting methods, such as interviews and classroom observations, to gain a better understanding of PMTs' approaches to addressing student errors. In addition, the present study focused on teachers' PCK. Future research may further explore the relationship between PMTs' SMK and PCK.

Statement about ethics

The content, purpose and process of this research were clarified to the Academic Ethics Committee of Shanghai Normal University, and this research was approved by the Academic Ethics Committee with approval number 2023004 (Ethical Approval of Shanghai Normal University). The purpose of this research was clarified to those involved, and they were informed that the data collected would be used for research purposes only. Those involved consented to participate in the study.

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Declaration of competing interest

No potential conflict of interest was reported by the authors.

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Data availability

Data will be made available on request.

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