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Research paper

Bifurcation of limit cycles near heteroclinic loops in near-Hamiltonian systems

Wei Geng, Yun Tian*

Department of Mathematics,Shanghai Normal University, Shanghai 200234, PR China

A R T I C L E I N F O

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A B S T R A C T

In this paper, we study the bifurcation of limit cycles near a heterocilinc loop with hyperbolic saddles in a perturbed planar Hamiltonian system. We present a method for computing the coefficients in the corresponding expansion of the first order Melnikov function. With more those coefficients, more limit cycles could be determined around the heteroclinic loop. An example of studying limit cycles produced from a heteroclinic loop with 2 saddles is investigated to illustrate our method.

1. Introduction

It is well-known that in 1900 Hilbert proposed a list of 23 mathematical problems at the Second International Congress of Mathematicians in Paris for being solved during the 20th century [13]. After 120 years, the 16th problem of the list is still open. The second part of Hilbert's 16th problem is to estimate the maximum number $H(n)$ of limit cycles in planar polynomial differential systems of degree n, and to investigate their distributions. It is still unknown whether $H(n)$ exists or not for $n \geq 2$. For more informations about the research progress of Hilbert's 16th problem, see review articles [14,16] and the references cited therein.

One restricted version of the Hilbert's 16th problem is to study the number of limit cycles in the following near-Hamiltonian system

$$\dot{x} = H_y(x, y) + \varepsilon f(x, y, \delta), \quad \dot{y} = -H_x(x, y) + \varepsilon g(x, y, \delta), \tag{1.1}$$

where $H(x, y)$, $f(x, y, \delta)$ and $g(x, y, \delta)$ are real polynomials in $(x, y) \in \mathbb{R}^2$, $\varepsilon > 0$ is small, $\delta \in D \subset \mathbb{R}^m$ is a vector parameter with D compact.

One important method for the study of limit cycle bifurcations in system (1.1) is the Abelian integral or Melnikov function given by

$$M(h, \delta) = \oint_{L_h} g\mathrm{d}x - f\mathrm{d}y, \tag{1.2}$$

where L_h is a continuous family of closed ovals defined by $H(x, y) = h$, $h_c < h < h_s$. If $M(h, \delta) \not\equiv 0$, the number of limit cycles in system (1.1) can be estimated by the the number of isolated zeros of $M(h, \delta)$ in h. The problem of studying zeros

* Corresponding author.
E-mail address: ytian22@shnu.edu.cn (Y. Tian).

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of $M(h, \delta)$ is called the weak Hilbert's 16th problem, which was posed by Arnold in [1]. For the applicaiton of Melnikov function to study the number of limit cycles, see survey papers [11,15] or books [4,12].

Under small perturbations in (1.1), limit cycles could be produced not only from the family of periodic orbits $\{L_h : h_c < h < h_s\}$, but aslo from its boundaries, which could be a center or homoclinic/heteroclinic loops defined by $H(x, y) = h_c$ or $H(x, y) = h_s$. The number of limit cycles bifurcating from a center or homoclinic/heteroclinic loops can be investigated by finding the zeros of $M(h, \delta)$ in the interval (h_c, h_s) near its endpoints through the asymptotic expansions of $M(h, \delta)$ at $h = h_c$ or $h = h_s$.

There are lots of research results on bifurcation of small limit cycles from centers, for instance see [3,7,10,19–21,23]. The Melnikov function $M(h, \delta)$ is analytic near an elementary center. In [10], a computationally efficient algorithm is provided for the computation of the coefficients of the asymptotic expansion of $M(h, \delta)$ near an elementary center. Another method equivalent to expanding $M(h, \delta)$ near an elementary center is using the linear terms of focus values, which was developed in [3]. Analysis on higher order terms of focus values was also introduced in [7,19] for finding the bifurcation of limit cycles from a center.

The asymptotic expansions of $M(h, \delta)$ play an important role in the study of bifurcations of limit cycles near heteroclinic loops, for instance see [2,9,17,18] and the references cited therein. For heteroclinic loops through hyperbolic saddles, in the literature we only find the formulas of the first four coefficients for the corresponding asymptotic expansion of $M(h, \delta)$, which were obtained in [9]. The expansions of $M(h, \delta)$ near heteroclinic loops through a saddle and a cusp were investigated in [2,17,18].

In this paper, we mainly focus on the study of limit cycles produced around heteroclinic loops through hyperbolic saddles. We shall present an approach for the computation of more coefficients in the corresponding asymptotic expansion of $M(h, \delta)$. Moreover, in order to find more limit cycles in (1.1), we shall investigate the bifurcation of limit cycles near a heteroclinic loop together with the bifurcation of small-amplitude limit cycles from an elementary center.

We suppose that the Hamiltonian system

$$\dot{x} = H_y(x, y), \quad \dot{y} = -H_x(x, y) \tag{1.3}$$

has a family of periodic orbits L_h defined by $H(x, y) = h,\ h \in (h_c, h_s) \triangleq J$, with two boundaries: an elementary center $C = (x_0, y_0)$ as the inner boundary and a heteroclinic loop L_s as the outer boundary. The heteroclinic loop L_s consists of n hyperbolic saddles $S_i = (x_i, y_i),\ i = 1, 2, \dots, n$, and n heteroclinic orbits $L_i,\ i = 1, 2, \dots, n$, connecting them. Suppose $h_c = H(x_0, y_0)$ and L_s is defined by $H(x, y) = h_s$.

Then for system (1.1), $M(h, \delta)$ has an expansion of the form at $h = h_c$

$$M(h, \delta) = \sum_{j\ge 0} b_j(\delta)(h - h_c)^{j+1}, \quad 0 \le h - h_c \ll 1, \tag{1.4}$$

with

$$b_0(\delta) = T_0\bar{b}_0(\delta), \quad \bar{b}_0(\delta) = (f_x + g_y)(C, \delta), \tag{1.5}$$

where $T_0 > 0$ is a constant. Theoretically, the maximum number of small limit cycles bifurcating from the center C can be determined by the study of the independence of the coefficients b_j's in (1.4). The asymptotic expansion of $M(h, \delta)$ at $h = h_s$ has the following form

$$\begin{aligned} M(h, \delta) &= \sum_{j\ge 0} (c_{2j}(\delta) + c_{2j+1}(\delta)(h - h_s)\ln|h - h_s|)(h - h_s)^j \\ &= c_0(\delta) + c_1(\delta)(h - h_s)\ln|h - h_s| + c_2(\delta)(h - h_s) + c_3(\delta)(h - h_s)^2\ln|h - h_s| + \dots, \quad 0 < h_s - h \ll 1. \end{aligned} \tag{1.6}$$

This paper is organized as follows. In Section 2, we shall present our method to compute the coefficients c_j's of the expansion of $M(h, \delta)$ in (1.6). Then using more coefficients c_j's in (1.6) and the coefficients b_j's in (1.4), we give a way to find more limit cycles for (1.1). In Section 3, an example is provided to illustrate our method.

2. Main results

Before we present our method for computing the coefficients of the asymptotic expansion of $M(h, \delta)$ in (1.6), we first give the formulas for its first four coefficients, which were presented in [9].

From Han et al. [9], for $c_0(\delta)$ and $c_1(\delta)$ in (1.6) we have

$$c_0(\delta) = \sum_{i=1}^{n} \int_{L_i} g\mathrm{d}x - f\mathrm{d}y, \tag{2.1}$$

$$c_1(\delta) = \sum_{i=1}^{n} c_1(S_i, \delta), \ \text{with } c_1(S_i, \delta) = -\frac{1}{|\lambda_i|}(f_x + g_y)(S_i, \delta), \tag{2.2}$$

where $\pm\lambda_i$ are the real eigenvalues of the linearized system of (1.3) at S_i. When $c_1(S_i,\delta)=0,\ i=1,\dots,n,\ c_2(\delta)$ and $c_3(\delta)$ are given by

$$c_2(\delta)=\oint_{L_s}(f_x+g_y)\mathrm{d}t=\sum_{i=1}^{n}\int_{L_i}(f_x+g_y)\mathrm{d}t, \tag{2.3}$$

$$c_3(\delta)=\sum_{i=1}^{n}c_3(S_i,\delta), \tag{2.4}$$

with

$$c_3(S_i,\delta)=\frac{-1}{2|\lambda_i|\lambda_i}\left\{(-3a_{30}-b_{21}+a_{12}+3b_{03})-\frac{1}{\lambda_i}[(2b_{02}+a_{11})(3h_{03}-h_{21})+(2a_{20}+b_{11})(3h_{30}-h_{12})]\right\}, \tag{2.5}$$

where h_{ij}, a_{ij} and b_{ij} are the coefficients of the series expansion of the following three new functions

$$\widetilde{H}(u,v)=h_s+\frac{\lambda_i}{2}(v^2-u^2)+\sum_{i+j\ge 3}h_{ij}u^iv^j, \tag{2.6}$$

$$\tilde{f}(u,v,\delta)=\sum_{i+j\ge 0}a_{ij}u^iv^j,\quad \tilde{g}(u,v,\delta)=\sum_{i+j\ge 0}b_{ij}u^iv^j, \tag{2.7}$$

which are obtained from $\widetilde{H}(u,v)=H(x,y)$ and

$$\tilde{f}(u,v,\delta)=t_{22}f(x,y,\delta)-t_{12}g(x,y,\delta),\quad \tilde{g}(u,v,\delta)=-t_{21}f(x,y,\delta)+t_{11}g(x,y,\delta).$$

By substituting the variable change

$$x=t_{11}u+t_{12}v+x_i,\quad y=t_{21}u+t_{22}v+y_i, \tag{2.8}$$

with $t_{11}t_{22}-t_{12}t_{21}=1$. By (2.8), system (1.1) is transformed into

$$\dot{u}=\widetilde{H}_v(u,v)+\varepsilon\tilde{f}(u,v,\delta),\quad \dot{v}=-\widetilde{H}_u(u,v)+\varepsilon\tilde{g}(u,v,\delta),$$

where $\widetilde{H}(u,v)$, $\tilde{f}(u,v,\delta)$ and $\tilde{g}(u,v,\delta)$ are given by (2.6) and (2.7).

Theorem 2.1. *For system (1.1), we assume $b_0=0$ and $c_1(S_i,\delta)=0,\ i=1,\dots,n$ in (1.4) and (1.6). Further suppose there exist analytic functions $P(x,y,\delta)$ and $Q(x,y,\delta)$ satisfying the following equation*

$$f_x+g_y=H_x(x,y)P(x,y,\delta)+H_y(x,y)Q(x,y,\delta),\quad (x,y)\in U\triangleq\bigcup_{h_c\le h\le h_s}L_h. \tag{2.9}$$

Then we have

$$\begin{aligned}&b_1=\frac{T_0}{2}\bar{b}_1,\quad \bar{b}_1=(P_x+Q_y)(C,\delta),\\&c_3(\delta)=\frac{1}{2}\sum_{i=1}^{n}\tilde{c}_1(S_i,\delta),\quad \tilde{c}_1(S_i,\delta)=-\frac{1}{|\lambda_i|}(P_x+Q_y)(S_i,\delta).\end{aligned} \tag{2.10}$$

When $\tilde{c}_1(S_i,\delta)=0,\ i=1,\dots,n,$ we get

$$\begin{aligned}&c_4(\delta)=\frac{1}{2}\oint_{L_s}(P_x+Q_y)\mathrm{d}t=\frac{1}{2}\sum_{i=1}^{n}\int_{L_i}(P_x+Q_y)\mathrm{d}t,\\&c_5(\delta)=\frac{1}{3}\sum_{i=1}^{n}\tilde{c}_3(S_i,\delta),\end{aligned} \tag{2.11}$$

where

$$\tilde{c}_3(S_i,\delta)=\frac{-1}{2|\lambda_i|\lambda_i}\left\{(-3\tilde{a}_{30}-\tilde{b}_{21}+\tilde{a}_{12}+3\tilde{b}_{03})-\frac{1}{\lambda_i}[(2\tilde{b}_{02}+\tilde{a}_{11})(3h_{03}-h_{21})+(2\tilde{a}_{20}+\tilde{b}_{11})(3h_{30}-h_{12})]\right\}, \tag{2.12}$$

if by the variable transformation (2.8), $\widetilde{H}(u,v)=H(x,y)$ satisfies (2.6), and

$$\begin{aligned}&\tilde{f}(u,v)=t_{22}P(x,y,\delta)-t_{12}Q(x,y,\delta)=\sum_{i+j\ge 0}\tilde{a}_{ij}u^iv^j,\\&\tilde{g}(u,v)=-t_{21}P(x,y,\delta)+t_{11}Q(x,y,\delta)=\sum_{i+j\ge 0}\tilde{b}_{ij}u^iv^j.\end{aligned}$$

Proof. Since $b_0 = 0$ and $c_1(S_i, \delta) = 0,\ i = 1, \ldots, n$, by (1.4) and (1.6), it is easy to get

$$\frac{\partial M}{\partial h} = 2b_1(h - h_c) + 3b_2(h - h_c)^2 + 4b_3(h - h_c)^3 + \cdots \tag{2.13}$$

for $0 \le h - h_c \ll 1$, and

$$\frac{\partial M}{\partial h} = c_2 + 2c_3(h - h_s)\ln|h - h_s| + (c_3 + 2c_4)(h - h_s) + 3c_5(h - h_s)^2 \ln|h - h_s| + \cdots \tag{2.14}$$

for $0 < h_s - h \ll 1$.

It is proved in [8] that for (1.2) the partial derivative of $M(h, \delta)$ with respect to h has the following form

$$\frac{\partial M}{\partial h}(h, \delta) = \oint_{L_h} (f_x + g_y)\mathrm{d}t. \tag{2.15}$$

Then by (2.9) and (2.15) we have

$$\begin{aligned}\frac{\partial M}{\partial h} &= \oint_{L_h} (H_x(x, y)P(x, y, \delta) + H_y(x, y)Q(x, y, \delta))\mathrm{d}t \\ &= \oint_{L_h} Q(x, y, \delta)\mathrm{d}x - P(x, y, \delta)\mathrm{d}y \triangleq \widetilde{M}(h, \delta).\end{aligned} \tag{2.16}$$

Expanding the new Melnikov function $\widetilde{M}(h, \delta)$ yields

$$\widetilde{M}(h, \delta) = \tilde{b}_0(\delta)(h - h_c) + \tilde{b}_1(\delta)(h - h_c)^2 + \cdots \tag{2.17}$$

for $0 \le h - h_c \ll 1$, and

$$\widetilde{M}(h) = \tilde{c}_0(\delta) + \tilde{c}_1(\delta)(h - h_s)\ln|h - h_s| + \tilde{c}_2(\delta)(h - h_s) + \tilde{c}_3(\delta)(h - h_s)^2 \ln|h - h_s| + \cdots \tag{2.18}$$

for $0 < h_s - h \ll 1$. Comparing the two expansions of $\widetilde{M}(h, \delta)$ above with (2.13) and (2.14) respectively, one obtains

$$b_1 = \tilde{b}_0/2, \quad c_2 = \tilde{c}_0, \quad c_3 = \tilde{c}_1/2, \quad c_4 = (\tilde{c}_2 - c_3)/2, \quad c_5 = \tilde{c}_3/3.$$

Then we can derive (2.10), (2.11), (2.12) by using (1.5), (2.2), (2.3), (2.4) and (2.5) to compute the coefficients $\tilde{b}_0,\ \tilde{c}_j,\ j = 0, 1, 2, 3$ for $\widetilde{M}(h, \delta)$. The proof is completed. □

Note that from (2.16) the derivative function $\partial M/\partial h$ can be written in the form of Melnikov function $\widetilde{M}(h)$ when (2.9) holds. If we can use Theorem 2.1 to compute the coefficients $\tilde{c}_4$ and $\tilde{c}_5$ for $\widetilde{M}(h)$, then we can derive formulas for the coefficients c_6 and c_7 for $M(h, \delta)$ by comparing the expansion (2.18) with (2.14).

Next, basing on this idea we shall present the formulas of c_6 and c_7 in (1.6). For $\widetilde{M}(h)$, by Theorem 2.1 we assume $\tilde{b}_0 = \tilde{c}_1(S_j, \delta) = 0,\ j = 1, \ldots, n$, and that there exist analytic functions $P_1(x, y, \delta)$ and $Q_1(x, y, \delta)$ such that

$$P_x(x, y, \delta) + Q_y(x, y, \delta) = H_x(x, y)P_1(x, y, \delta) + H_y(x, y)Q_1(x, y, \delta), \quad (x, y) \in U. \tag{2.19}$$

Then by Theorem 2.1 we derive

$$\begin{aligned}\tilde{b}_1 &= \frac{T_0}{2}(P_{1x} + Q_{1y})(C, \delta), \\ \tilde{c}_3(\delta) &= \frac{1}{2}\sum_{i=1}^{n} \hat{c}_1(S_i, \delta), \quad \hat{c}_1(S_i, \delta) = -\frac{1}{|\lambda_i|}(P_{1x} + Q_{1y})(S_i, \delta).\end{aligned} \tag{2.20}$$

When $\hat{c}_1(S_i, \delta) = 0,\ i = 1, \ldots, n$, for $\tilde{c}_4$ and $\tilde{c}_5$ we get

$$\begin{aligned}\tilde{c}_4(\delta) &= \oint_{L_s} (P_{1x} + Q_{1y})\mathrm{d}t = \sum_{i=1}^{n} \int_{L_i} (P_{1x} + Q_{1y})\mathrm{d}t, \\ \tilde{c}_5(\delta) &= \frac{1}{3}\sum_{i=1}^{n} \hat{c}_3(S_i, \delta),\end{aligned} \tag{2.21}$$

with

$$\hat{c}_3(S_i, \delta) = \frac{-1}{2|\lambda_i|\lambda_i}\left\{(-3\hat{a}_{30} - \hat{b}_{21} + \hat{a}_{12} + 3\hat{b}_{03}) - \frac{1}{\lambda_i}[(2\hat{b}_{02} + \hat{a}_{11})(3h_{03} - h_{21}) + (2\hat{a}_{20} + \hat{b}_{11})(3h_{30} - h_{12})]\right\},$$

where $\hat{c}_3(S_i, \delta)$ is obtained by computing $\tilde{c}_3(S_i, \delta)$ with respect to $H(x, y)$, $P_1(x, y, \delta)$ and $Q_1(x, y, \delta)$ by the variable transformation (2.8), and we have $\widetilde{H}(u, v) = H(x, y)$ satisfying (2.6) and

$$\begin{aligned}t_{22}P_1(x, y, \delta) - t_{12}Q_1(x, y, \delta) &= \sum_{i+j\ge 0} \hat{a}_{ij}u^i v^j, \\ -t_{21}P_1(x, y, \delta) + t_{22}Q_1(x, y, \delta) &= \sum_{i+j\ge 0} \hat{b}_{ij}u^i v^j.\end{aligned}$$

Comparing the two expansions of $\widetilde{M}(h,\delta)$ in (2.17) and (2.18) with (2.13) and (2.14) respectively, one obtains

$$3b_2=\tilde{b}_1,\quad c_5+3c_6=\tilde{c}_4,\quad 4c_7=\tilde{c}_5.$$

Then we have the following theorem.

Theorem 2.2. *Suppose for system (1.1), when $b_0=b_1=0,\ c_1(S_i,\delta)=\tilde{c}_1(S_i,\delta)=0,\ i=1,\dots,n,$ there exist analytic functions $P(x,y,\delta),\ Q(x,y,\delta),\ P_1(x,y,\delta)$ and $Q_1(x,y,\delta)$ satisfying (2.9) and (2.19). Then we have*

$$b_2=\frac{1}{3}\tilde{b}_1,\quad c_5=\frac{1}{3}\tilde{c}_3,\quad c_6=\frac{1}{3}\tilde{c}_4|_{\hat{c}_1(S_i,\delta)=0},\quad c_7=\frac{1}{4}\tilde{c}_5|_{\hat{c}_1(S_i,\delta)=0},$$

where $\tilde{b}_1,\ \tilde{c}_3,\ \tilde{c}_4$ and $\tilde{c}_5$ are given in (2.20) and (2.21).

By computing more coefficients in (1.4) and (1.6), we can find more zeros of $M(h,\delta)$ in h near the endpoints of the interval J, which imply more limit cycles produced around the center C and the heteroclinic loop L_s for system (1.1). We use (i,j,k) distribution to represent i small-amplitude limit cycles bifurcating from the center C, k limit cycles bifurcating from the heteroclinic loop L_s, and j limit cycles produced between these two groups of limit cycles for system (1.1).

Let $[x]$ denote the integer part of x. For finding limit cycles in system (1.1), we have the following theorem.

Theorem 2.3. *If for some $\delta_0\in\mathbb{R}^m$ and two integers k_1 and k_2 we have*

$$\begin{aligned}&b_i(\delta_0)=0,\ i=0,1,\dots,k_1-1,\quad c_j(\delta_0)=0,\quad j=0,1,\dots,k_2-1,\\&\text{rank}\frac{\partial(b_0,b_1,\dots,b_{k_1-1},c_0,c_1,\dots,c_{k_2-1})}{\partial(\delta_1,\dots,\delta_m)}(\delta_0)=k_1+k_2,\end{aligned}\tag{2.22}$$

and

$$\mu\triangleq(-1)^{\xi(k_2)}b_{k_1}(\delta_0)c_{k_2}(\delta_0)\neq0,\ \text{where}\ \xi(k_2)=\left[\frac{\text{irem}(k_2,4)}{2}\right],$$

and the function irem$(k_2,4)$ *computes the integer remainder of k_2 divided by 4, then the system (1.1) has k_1+k_2+1 (or k_1+k_2) limit cycles as $\mu<0$ (or >0) for some (ε,δ) near $(0,\delta_0)$, with $(k_1,1,k_2)$ distribution (or $(k_1,0,k_2)$ distribution).*

Proof. Note that when $0<h_s-h\ll1$, for the integer $k_2=4k+l$, where $l=\text{irem}(k_2,4)$, the term with the coefficient c_{k_2} in the asymptotic expansion of $M(h,\delta)$ (1.6) has

$$\begin{cases}(h-h_s)^{2k}>0,\ \text{if}\ l=0;\\(h-h_s)^{2k+1}\ln|h-h_s|>0,\ \text{if}\ l=1;\\(h-h_s)^{2k+1}<0,\ \text{if}\ l=2;\\(h-h_s)^{2k+2}\ln|h-h_s|<0,\ \text{if}\ l=3.\end{cases}$$

Here we only show the proof for the case of $l=1$ with $b_{k_1}(\delta_0)<0,\ c_{k_2}(\delta_0)>0$. Then $\xi(k_2)=0$ and $\mu=b_{k_1}(\delta_0)c_{k_2}(\delta_0)<0$. The other cases can be similarly proved.

In this case, from (1.4) and (1.6) we have

$$M(h,\delta_0)=b_{k_1}(\delta_0)(h-h_c)^{k_1+1}+\cdots<0$$

for $0<h-h_c\ll1$, and

$$M(h,\delta_0)=c_{k_2}(\delta_0)(h-h_s)^{2k+1}\ln|h-h_s|+\cdots>0$$

for $0<h_s-h\ll1$. Then the function $M(h,\delta)$ has at least one zero $h_0(\delta)\in J$ in h, having an odd multiplicity, for δ near δ_0.

By (2.22) we can take $b_0,\ b_1,\ \dots,\ b_{k_1-1}$, and $c_0,\ c_1,\ \dots,\ c_{k_2-1}$ as free parameters for δ near δ_0. We vary the values of $c_{k_2-1},\ c_{k_2-2},\ \dots,\ c_1$ and c_0 in turn such that

$$1\gg-c_{k_2-1}\gg c_{k_2-2}\gg\cdots\gg c_{4j+3}\gg c_{4j+2}\gg-c_{4j+1}\gg-c_{4j}\cdots\gg-c_1\gg-c_0>0.$$

By this way, we can change the sign of $M(h,\delta)$ five times for $0<h_s-h\ll1$. Then we get k_2 simple zeros $h_i(\delta),\ i=1,2,\dots,k_2$, of $M(h,\delta)$ near h_s for δ near δ_0, with the zero $h_0(\delta)$ of $M(h,\delta)$ still existing.

Similarly, we can vary the values of $b_{k_1-1},\ b_{k_1-2},\ \dots,\ b_1$ and b_0 in turn such that

$$1\gg b_{k_1-1}\gg-b_{k_1-2}\gg b_{k_1-3}\gg-b_{k_1-4}\gg\cdots>0.$$

Then the sign of $M(h,\delta)$ is changed k_1 times for $0<h-h_c\ll1$, and $M(h,\delta)$ can have k_1 simple zeros near $h=h_c$ for $\|\delta-\delta_0\|\ll1$, with the zeros $h_i(\delta),\ 0\le i\le k_2$, still existing.

Therefore, when $\mu<0$ and irem$(k_2,4)=1$, for δ near δ_0 system (1.1) can have k_1 small-amplitude limit cycles appearing around the center C, k_2 limit cycles appearing near the heteroclinic loop L_s, and another one limit cycle between these two groups of limit cycles. This completes the proof. □

3. Limit cycles near a two-saddle loop

The number of limit cycles bifurcating from a heteroclinic loop with two hyperbolic saddles (two-saddle loop) is investigated in [6,22,24,5]. Gavrilov and Iliev [6] proved that the cyclicity of a two-saddle loop in a perturbed quadratic Hamiltonian system is less than or equal to 3. As an application of Theorems 2.1, 2.2 and 2.3, we shall study the bifurcation of limit cycles near a two-saddle loop in a cubic Hamiltonian system under small perturbations.

Consider the following near-Hamiltonian system

$$\dot{x} = y + \varepsilon f(x, y, \delta), \quad \dot{y} = -x + x^3 + \varepsilon g(x, y, \delta), \tag{3.1}$$

where

$$f(x, y, \delta) = \sum_{i+j=0}^{n} a_{ij} x^i y^j, \quad g(x, y, \delta) = \sum_{i+j=0}^{n} b_{ij} x^i y^j, \tag{3.2}$$

and δ is a parameter vector consisting of all the coefficients a_{ij}'s and b_{ij}'s in (3.2).

The Hamiltonian function is given by

$$H(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{4}x^4. \tag{3.3}$$

It is easy to see that the corresponding Hamiltonian system has an elementary center $C = (0, 0)$ and two hyperbolic saddles $S_1 = (-1, 0)$, $S_2 = (1, 0)$, with two heteroclinic orbits L_1 and L_2 connecting them, where

$$L_1 : y = \frac{1}{\sqrt{2}}(1 - x^2), \quad L_2 : y = -\frac{1}{\sqrt{2}}(1 - x^2), \quad |x| < 1.$$

The family of periodic orbits around the center C are given by $L_h \subset \{(x, y) \in H(x, y) = h, \ h \in (0, \frac{1}{4})\}$ with the heteroclinic loop $L_s = L_1 \cup L_2 \cup \{S_1, S_2\}$ as the outer boundary.

For system (3.1), the Melnikov function $M(h)$ is given by

$$M(h, \delta) = \oint_{L_h} g(x, y, \delta)\mathrm{d}x - f(x, y, \delta)\mathrm{d}y. \tag{3.4}$$

In order to apply Theorems 2.1 and 2.2, we need to find analytic functions $P(x, y, \delta)$ and $Q(x, y, \delta)$ satisfying (2.9) for system (3.1). Let $F(x, y, \delta) = f_x(x, y, \delta) + g_y(x, y, \delta)$. Then for polynomial $F(x, y, \delta)$ with $F(C, \delta) = F(S_1, \delta) = F(S_2, \delta) = 0$, it is easy to verify that the following two polynomials

$$P(x, y, \delta) = \frac{F(x, 0)}{x - x^3}, \quad Q(x, y, \delta) = \frac{F(x, y) - F(x, 0)}{y}, \tag{3.5}$$

satisfy (2.9).

System (3.1) with some polynomial perturbations of degree 7 was studied in [22], where four limit cycles are found near the loop L_s including one alien limit cycle. Here because of the symmetry of the unperturbed system of (3.1), we study the number of limit cycles in (3.1) with general polynomial perturbations of degree $n = 5, 7, 9$. We have the following theorem.

Theorem 3.1. *With proper perturbations, system (3.1) can have 4 limit cycles with (1,0,3) distribution for $n = 5$; 6 limit cycles with (1,0,5) distribution for $n = 7$; 9 limit cycles with (2,1,6) distribution for $n = 9$.*

By (3.2) and (3.4), the Melnikov function $M(h)$ can be rewritten as

$$M(h, \delta) = \oint_{L_h} \sum_{i+j=0}^{n-1} \bar{b}_{ij} x^i y^{j+1} \mathrm{d}x, \ \text{where} \ \bar{b}_{ij} = \frac{i+1}{j+1} a_{i+1,j} + b_{i,j+1}. \tag{3.6}$$

By (3.3), all the periodic orbits L_h, $h \in (0, 1/4)$ are symmetric with respect to the x-axis and y-axis. Then we can get

$$\oint_{L_h} x^i y^{j+1} \mathrm{d}x = 0, \ \text{when either } i \text{ or } j \text{ is odd.}$$

Then $M(h, \delta)$ can be further simplified into the form

$$M(h, \delta) = \oint_{L_h} \bar{g}(x, y, \delta)\mathrm{d}x, \tag{3.7}$$

where

$$\bar{g}(x, y, \delta) = \sum_{0 \le i \le n}^{i \text{ even}} \sum_{1 \le j \le n-i}^{j \text{ odd}} \bar{b}_{ij} x^i y^j. \tag{3.8}$$

By (3.6), we can take the coefficients $\bar{b}_{ij}$'s as free parameters.

Note that $M(h,\delta)$ in (3.7) is the first order Melnikov function of the following near-Hamiltonian system

$$\dot{x} = H_y(x,y), \quad \dot{y} = -H_x(x,y) + \varepsilon \bar{g}(x,y), \tag{3.9}$$

with $H(x,y)$ given by (3.3). To prove Theorem 3.1, we only need to study the asymptotic expansions of $M(h,\delta)$ in (3.7) at $h=0$ and $h=1/4$.

Proof. Firstly, we consider the case of $n=5$. By (3.8) $\bar{g}(x,y,\delta)$ has the following form

$$\bar{g}(x,y,\delta) = \bar{b}_{0,1}y + \bar{b}_{2,1}x^2y + \bar{b}_{0,3}y^3 + \bar{b}_{4,1}x^4y + \bar{b}_{2,3}x^2y^3 + \bar{b}_{0,5}y^5.$$

For the Melnikov function (3.7), by (1.5) and (2.1) we obtain

$$\bar{b}_0(\delta) = -\bar{b}_{0,1}, \quad c_0(\delta) = \frac{4\sqrt{2}}{3465}(1155\bar{b}_{0,1} + 231\bar{b}_{2,1} + 396\bar{b}_{0,3} + 99\bar{b}_{4,1} + 44\bar{b}_{2,3} + 160\bar{b}_{0,5}). \tag{3.10}$$

Noting that the eigenvalues of the unperturbed system of (3.9) at both saddles $S_1=(-1,0)$ and $S_2=(1,0)$ are $\pm\sqrt{2}$, by (2.2) we have

$$c_1(S_1,\delta) = c_1(S_2,\delta) = -\frac{\sqrt{2}}{2}(\bar{b}_{0,1} + \bar{b}_{2,1} + \bar{b}_{4,1}),$$

and

$$c_1(\delta) = c_1(S_1,\delta) + c_1(S_2,\delta) = -\sqrt{2}(\bar{b}_{0,1} + \bar{b}_{2,1} + \bar{b}_{4,1}). \tag{3.11}$$

To compute $c_2(\delta)$ and $c_3(\delta)$, we take $c_1(S_1,\delta)=c_1(S_2,\delta)=0$. By (2.3) we obtain

$$\begin{aligned} c_2(\delta) &= \oint_{L_s} \bar{g}_y(x,y,\delta)\mathrm{d}t = \oint_{L_s} \frac{\bar{g}_y(x,y,\delta)}{y}\mathrm{d}x \\ &= \frac{4\sqrt{2}}{105}(-105\bar{b}_{2,1} + 105\bar{b}_{0,3} + 21\bar{b}_{2,3} - 140\bar{b}_{4,1} + 60\bar{b}_{0,5}). \end{aligned} \tag{3.12}$$

For $\bar{b}_1$, c_3 and c_4, we need to find $P(x,y,\delta)$ and $Q(x,y,\delta)$ satisfying

$$\bar{g}_y(x,y,\delta) = H_x(x,y)P(x,y,\delta) + H_y(x,y)Q(x,y,\delta). \tag{3.13}$$

Setting $\bar{b}_0(\delta)=c_1(\delta)=0$ yields $\bar{b}_{0,1}=0, \bar{b}_{2,1}=-\bar{b}_{4,1}$, under which by (3.5) we have

$$\begin{aligned} P(x,y,\delta) &= \frac{\bar{g}_y(x,0,\delta)}{x-x^3} = -\bar{b}_{4,1}x, \\ Q(x,y,\delta) &= \frac{\bar{g}_y(x,y,\delta) - \bar{g}_y(x,0,\delta)}{y} = 3\bar{b}_{0,3}y + 3\bar{b}_{2,3}x^2y + 5\bar{b}_{0,5}y^3. \end{aligned}$$

Then by Theorem 2.1, we have

$$\begin{aligned} \bar{b}_1(\delta) &= (P_x + Q_y)(C,\delta) = 3\bar{b}_{0,3} - \bar{b}_{4,1}, \\ c_3(\delta) &= \frac{1}{2}\sum_{i=1}^{2}\left[-\frac{1}{\sqrt{2}}(P_x+Q_y)(S_{i0},\delta)\right] = -\sqrt{2}\big(\bar{b}_{2,1} + 3\bar{b}_{0,3} + 3\bar{b}_{2,3}\big). \end{aligned} \tag{3.14}$$

By (3.10), (3.11), (3.12) and (3.14), solving $\bar{b}_0(\delta)=c_0(\delta)=c_1(\delta)=c_2(\delta)=0$ yields

$$\bar{b}_{0,1}=0, \quad \bar{b}_{2,1}=-\bar{b}_{4,1}, \quad \bar{b}_{0,3}=\frac{1}{3}\bar{b}_{4,1} - \frac{15}{77}\bar{b}_{0,5}, \quad \bar{b}_{2,3} = -\frac{145}{77}\bar{b}_{0,5},$$

and

$$\bar{b}_1(\delta) = -\frac{45}{77}\bar{b}_{0,5} \neq 0, \quad c_3(\delta) = \frac{120\sqrt{2}}{77}\bar{b}_{0,5} \neq 0,$$

when $b_{0,5}\neq 0$. Therefore, by Theorem 2.3 the system (3.1) can have three limit cycles near L_s and one small-amplitude limit cycle near C.

Next, we study the case of $n=7$. By (3.8) the polynomial $\bar{g}(x,y,\delta)$ is given by

$$\bar{g}(x,y,\delta) = \bar{b}_{0,1}y + \bar{b}_{2,1}x^2y + \bar{b}_{0,3}y^3 + \bar{b}_{4,1}x^4y + \bar{b}_{2,3}x^2y^3 + \bar{b}_{0,5}y^5 + \bar{b}_{6,1}x^6y + \bar{b}_{4,3}x^4y^3 + \bar{b}_{2,5}x^2y^5 + \bar{b}_{0,7}y^7.$$

By (1.5), (2.1) and (2.2) we have $\bar{b}_0=\bar{b}_{0,1}$, and

$$\begin{aligned} c_0(\delta) &= \frac{4\sqrt{2}}{45045}(15015\bar{b}_{0,1} + 3003\bar{b}_{2,1} + 5148\bar{b}_{0,3} + 572\bar{b}_{2,3} + 1287\bar{b}_{4,1} + 2080\bar{b}_{0,5} \\ &\quad + 160\bar{b}_{2,5} + 156\bar{b}_{4,3} + 715\bar{b}_{6,1} + 896\bar{b}_{0,7}), \\ c_1(\delta) &= c_1(S_1,\delta) + c_1(S_2,\delta) = -\sqrt{2}(\bar{b}_{0,1} + \bar{b}_{2,1} + \bar{b}_{4,1} + \bar{b}_{6,1}), \end{aligned}$$

where $c_1(S_1,\delta)=c_1(S_2,\delta)$. When $c_1(S_1,\delta)=c_1(S_2,\delta)=0$, by (2.3) we derive

$$c_2(\delta)=\frac{4\sqrt{2}}{3465}(-3465\bar{b}_{2,1}+3465\bar{b}_{0,3}+693\bar{b}_{2,3}-4620\bar{b}_{4,1}+1980\bar{b}_{0,5}+220\bar{b}_{2,5}+297\bar{b}_{4,3}-5313\bar{b}_{6,1}+1120\bar{b}_{0,7}).$$

Similarly, for $\bar{b}_1$, $c_3(\delta)$, $c_4(\delta)$ and $c_5(\delta)$, we need to compute $P(x,y,\delta)$ and $Q(x,y,\delta)$ satisfying (3.13). Setting $\bar{b}_0(\delta)=c_1(\delta)=0$ yields $\bar{b}_{0,1}=0$, $\bar{b}_{2,1}=-\bar{b}_{4,1}-\bar{b}_{6,1}$, under which by (3.5) we have

$$P(x,y,\delta)=-\big(\bar{b}_{4,1}+\bar{b}_{6,1}\big)x-\bar{b}_{6,1}x^3,$$
$$Q(x,y,\delta)=3\bar{b}_{0,3}y+3\bar{b}_{2,3}x^2y+5\bar{b}_{0,5}y^3+5\bar{b}_{2,5}x^2y^3+3\bar{b}_{4,3}x^4y+7\bar{b}_{0,7}y^5.$$

Then by Theorem 2.1, we get

$$\bar{b}_1(\delta)=3\bar{b}_{0,3}-\bar{b}_{4,1}-\bar{b}_{6,1},$$
$$c_3(\delta)=\tfrac{1}{2}(\tilde{c}_1(S_1,\delta)+\tilde{c}_1(S_2,\delta))=-\tfrac{\sqrt{2}}{2}(3\bar{b}_{0,3}-\bar{b}_{4,1}+3\bar{b}_{2,3}-4\bar{b}_{6,1}+3\bar{b}_{4,3}).$$

and $\tilde{c}_1(S_1,\delta)=\tilde{c}_1(S_2,\delta)$. When $\tilde{c}_1(S_1,\delta)=\tilde{c}_1(S_2,\delta)=0$, by (2.11)

$$c_4(\delta)=\frac{1}{2}\oint_{L_s}\frac{P_x+Q_y}{y}dx=-2\sqrt{2}(-3\bar{b}_{2,3}+5\bar{b}_{0,5}+3\bar{b}_{6,1}-4\bar{b}_{4,3}+\bar{b}_{2,5}+4\bar{b}_{0,7}).$$

To compute $c_5(\delta)$ by (2.11), for $\tilde{c}_3(S_1,\delta)$ we make the following variable transformation

$$u=y/\sqrt[4]{2},\quad v=-\sqrt[4]{2}(x+1),$$

and get

$$\widetilde{H}(u,v)=H\left(-\frac{v}{\sqrt[4]{2}}-1,\sqrt[4]{2}u\right)=\frac{1}{4}-\frac{\sqrt{2}}{2}(v^2-u^2)-\frac{2^{1/4}}{2}v^3-\frac{1}{8}v^4,$$

$$\begin{aligned}\tilde{f}(u,v)&=\frac{1}{\sqrt[4]{2}}Q\left(-\frac{v}{\sqrt[4]{2}}-1,\sqrt[4]{2}u\right)\\&=(\bar{b}_{4,1}+4\bar{b}_{6,1})u+3\sqrt[4]{8}(\bar{b}_{2,3}+2\bar{b}_{4,3})uv+5\sqrt{2}(\bar{b}_{0,5}+\bar{b}_{2,5})u^3\\&\quad+\frac{3}{\sqrt{2}}(\bar{b}_{2,3}+3\bar{b}_{4,3})uv^2+10\sqrt[4]{2}\bar{b}_{2,5}u^3v+6\sqrt[4]{2}\bar{b}_{4,3}uv^3+14\bar{b}_{0,7}u^5+5\bar{b}_{2,5}u^3v^2+\frac{3}{2}\bar{b}_{4,3}uv^4,\\ \tilde{g}(u,v)&=-\sqrt[4]{2}P\left(-\frac{v}{\sqrt[4]{2}}-1,\sqrt[4]{2}u\right)\\&=-\sqrt[4]{2}(\bar{b}_{4,1}+2\bar{b}_{6,1})-(\bar{b}_{4,1}+4\bar{b}_{6,1})v-\frac{3\sqrt[4]{8}}{2}\bar{b}_{6,1}v^2-\frac{\sqrt{2}}{2}\bar{b}_{6,1}v^3.\end{aligned}$$

Then by (2.12) we get

$$\tilde{c}_3(S_1,\delta)=-\frac{3\sqrt{2}}{4}(\bar{b}_{2,3}+5\bar{b}_{0,5}+5\bar{b}_{2,5}-\bar{b}_{6,1}).$$

Similarly, for $\tilde{c}_3(S_2,\delta)$ we make the following variable transformation

$$u=\frac{1}{\sqrt[4]{2}}y,\quad v=-\sqrt[4]{2}(x-1),$$

and derive $\tilde{c}_3(S_2,\delta)=\tilde{c}_3(S_1,\delta)$. Then by (2.11) we can have

$$c_5(\delta)=\frac{1}{3}(\tilde{c}_3(S_1,\delta)+\tilde{c}_3(S_2,\delta))=-\frac{\sqrt{2}}{2}(\bar{b}_{2,3}+5\bar{b}_{0,5}+5\bar{b}_{2,5}-\bar{b}_{6,1}).$$

Noting that $\bar{b}_0(\delta)=c_0(\delta)=c_1(\delta)=c_2(\delta)=c_3(\delta)=c_4(\delta)=0$ imply

$$\bar{b}_{0,1}=0,\quad \bar{b}_{2,1}=-\bar{b}_{4,1}-\bar{b}_{6,1},\quad \bar{b}_{0,3}=\frac{1}{3}\bar{b}_{4,1}+\frac{1}{3}\bar{b}_{61}+\frac{1}{11}\bar{b}_{07},$$
$$\bar{b}_{2,3}=\bar{b}_{6,1}-\bar{b}_{4,3}-\frac{1}{11}\bar{b}_{0,7},\quad \bar{b}_{0,5}=\frac{1}{5}\bar{b}_{4,3}-\frac{27}{55}\bar{b}_{0,7},\quad \bar{b}_{2,5}=-\frac{20}{11}\bar{b}_{0,7},$$

and

$$\bar{b}_1=\frac{3}{11}\bar{b}_{0,7}\neq 0,\quad c_5(\delta)=\frac{64\sqrt{2}}{11}\bar{b}_{0,7}\neq 0$$

when $\bar{b}_{0,7}\neq 0$. Therefore, by Theorem 2.3 system (3.1) can have five limits cycles near L_s and one small limits cycle near C when $n=7$.

Third, when $n = 9$, $\bar{g}(x, y, \delta)$ has the following form

$$\bar{g}(x,y,\delta) = \bar{b}_{0,1}y + \bar{b}_{2,1}x^2y + \bar{b}_{0,3}y^3 + \bar{b}_{4,1}x^4y + \bar{b}_{2,3}x^2y^3 + \bar{b}_{0,5}y^5 + \bar{b}_{6,1}x^6y + \bar{b}_{4,3}x^4y^3 + \bar{b}_{2,5}x^2y^5 + \bar{b}_{0,7}y^7 + \bar{b}_{8,1}x^8y + \bar{b}_{6,3}x^6y^3 + \bar{b}_{4,5}x^4y^5 + \bar{b}_{2,7}x^2y^7 + \bar{b}_{0,9}y^9.$$

Then, we also have

$$\bar{b}_0(\delta) = \bar{b}_{0,1}, \quad c_1(\delta) = -\sqrt{2}(\bar{b}_{0,1} + \bar{b}_{2,1} + \bar{b}_{4,1} + \bar{b}_{6,1} + \bar{b}_{8,1}),$$

$$c_0(\delta) = \frac{4\sqrt{2}}{14549535}(4849845\bar{b}_{0,1} + 969969\bar{b}_{2,1} + 1662804\bar{b}_{0,3} + 184756\bar{b}_{2,3} + 415701\bar{b}_{4,1} + 671840\bar{b}_{0,5} + 230945\bar{b}_{6,1} + 50388\bar{b}_{4,3} + 51680\bar{b}_{2,5} + 289408\bar{b}_{0,7} + 146965\bar{b}_{8,1} + 19380\bar{b}_{6,3} + 10336\bar{b}_{4,5} + 17024\bar{b}_{2,7} + 129024\bar{b}_{0,9}),$$

with $c_1(S_1, \delta) = c_1(S_2, \delta)$. When $c_1(S_1, \delta) = 0$, we get

$$c_2(\delta) = \frac{4\sqrt{2}}{45045}(-45045\bar{b}_{2,1} + 45045\bar{b}_{0,3} - 60060\bar{b}_{4,1} + 9009\bar{b}_{2,3} + 25740\bar{b}_{0,5} - 69069\bar{b}_{6,1} + 3861\bar{b}_{4,3} + 2860\bar{b}_{2,5} + 14560\bar{b}_{0,7} - 75504\bar{b}_{8,1} + 2145\bar{b}_{6,3} + 780\bar{b}_{4,5} + 1120\bar{b}_{2,7} + 8064\bar{b}_{0,9}),$$

$$c_3(\delta) = -\frac{\sqrt{2}}{2}(\bar{b}_{2,1} + 3\bar{b}_{0,3} + 3\bar{b}_{2,3} - 3\bar{b}_{6,1} + 3\bar{b}_{4,3} - 8\bar{b}_{8,1} + 3\bar{b}_{6,3}).$$

We can also obtain $c_4(\delta)$, $c_5(\delta)$, $c_6(\delta)$ and $c_7(\delta)$ by calculating $P(x, y, \delta)$, $Q(x, y, \delta)$, $P_1(x, y, \delta)$ and $Q_1(x, y, \delta)$ by Theorems 2.1 and 2.2. By Theorem 2.1, when $\bar{b}_0 = c_1(S_1, \delta) = 0$, we obtain

$$\bar{b}_1(\delta) = 3\bar{b}_{0,3} - \bar{b}_{4,1} - \bar{b}_{6,1} - \bar{b}_{8,1},$$

$$c_4(\delta) = \frac{2\sqrt{2}}{3465}(-10395\bar{b}_{2,3} + 17325\bar{b}_{0,5} + 10395\bar{b}_{6,1} - 13860\bar{b}_{4,3} + 3465\bar{b}_{2,5} + 13860\bar{b}_{0,7} + 33495\bar{b}_{8,1} - 15939\bar{b}_{6,3} + 1485\bar{b}_{4,5} + 1540\bar{b}_{2,7} + 10080\bar{b}_{0,9}),$$

$$c_5(\delta) = -\frac{\sqrt{2}}{2}(\bar{b}_{2,3} + 5\bar{b}_{0,5} - \bar{b}_{6,1} + 5\bar{b}_{2,5} - \bar{b}_{8,1} - 3\bar{b}_{6,3} + 5\bar{b}_{4,5}),$$

with $\tilde{c}_1(S_1, \delta) = 0$. By Theorem 2.2, when further $\bar{b}_1 = \tilde{c}_1(S_1, \delta) = 0$, we derive

$$\bar{b}_2(\delta) = \frac{3}{8}(5\bar{b}_{2,3} + 120\bar{b}_{0,5} - 3\bar{b}_{4,3} - 5\bar{b}_{6,1} - 3\bar{b}_{6,3}),$$

$$c_6(\delta) = \frac{2\sqrt{2}}{3}\left(-15\bar{b}_{2,5} + 35\bar{b}_{0,7} + 9\bar{b}_{6,3} - 20\bar{b}_{4,5} + 7\bar{b}_{2,7} + 36\bar{b}_{0,9}\right),$$

$$c_7(\delta) = -\frac{\sqrt{2}}{8}(5\bar{b}_{2,5} + 35\bar{b}_{0,7} - 3\bar{b}_{6,3} + 35\bar{b}_{2,7}).$$

Solving $\bar{b}_0(\delta) = \bar{b}_1(\delta) = c_0(\delta) = c_1(\delta) = c_2(\delta) = c_3(\delta) = c_4(\delta) = c_5(\delta) = 0$ implies

$$\bar{b}_{0,1} = 0, \quad \bar{b}_{2,1} = -\bar{b}_{4,1} - \bar{b}_{6,1} - \bar{b}_{8,1}, \quad \bar{b}_{0,3} = \frac{1}{3}(\bar{b}_{4,1} + \bar{b}_{6,1} + \bar{b}_{8,1}),$$

$$\bar{b}_{2,3} = -5\bar{b}_{0,5} + \bar{b}_{6,1} + \bar{b}_{8,1} + \frac{63}{209}\bar{b}_{0,9}, \quad \bar{b}_{4,3} = 5\bar{b}_{0,5} + \frac{5}{3}\bar{b}_{8,1} - \bar{b}_{6,3} - \frac{63}{209}\bar{b}_{0,9},$$

$$\bar{b}_{2,5} = \frac{3}{5}\bar{b}_{6,3} - \bar{b}_{4,5} - \frac{63}{1045}\bar{b}_{0,9}, \quad \bar{b}_{0,7} = \frac{1}{7}\bar{b}_{4,5} - \frac{351}{1045}\bar{b}_{0,9}, \quad \bar{b}_{2,7} = -\frac{4248}{1045}\bar{b}_{0,9},$$

and

$$\bar{b}_2 = \frac{189}{209}\bar{b}_{0,9} \neq 0, \quad c_6(\delta) = -\frac{2304\sqrt{2}}{1045}\bar{b}_{0,9} \neq 0,$$

when $\bar{b}_{0,9} \neq 0$. Then by Theorem 2.3, system (3.1) can have six limit cycles near L_s, two small limit cycles near C, and another one limit cycle between them when $n = 9$. This completes the proof. □

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Wei Geng: Investigation, Resources, Writing - original draft. **Yun Tian:** Conceptualization, Methodology, Writing - review & editing, Supervision.

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