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Research paper

Dissipativity of variable-stepsize Runge-Kutta methods for nonlinear functional differential equations with application to Nicholson's blowflies models☆

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ABSTRACT

This paper deals with dissipativity of the variable-stepsize Runge-Kutta methods applied to nonlinear Volterra functional differential equations in Hilbert spaces. The conditional dissipativity of variable-stepsize Runge-Kutta methods is analyzed and hence some new numerical dissipative criteria are derived. The resulting dissipative criteria extend and improve the existing results. Especially, for the algebraically stable variable-stepsize Runge-Kutta methods, we obtain a sharper dissipative result. In the end, we apply some concrete variable-stepsize Runge-Kutta methods to the three classes of nonlinear Nicholson's blowflies models. The presented numerical examples further illustrate the theoretical results.

1. Introduction

In this paper, we study the long time behavior of variable stpesize Runge-Kutta (RK) methods for Volterra functional differential equations (VFDEs, or rewritten as Retarded functional differential equations, RFDEs)

$$\begin{cases} y'(t) = f(t, y(t), y(\cdot)), & t \geq 0, \\ y(t) = \phi(t), & -\tau \leq t \leq 0, \end{cases} \tag{1.1}$$

where τ is a positive constant, ϕ is a given initial function. Equations of this type arise in many applications such as control theory, population dynamics, heat conduction in materials with thermal memory, biosciences, and so on (see [1,25,64], and references therein). Especially, Nicholson blowflies models have been extensively used to model various biological phenomena including production of blood cells (see, e.g., [4,6,16,32,33,39,42,43]). Much work has been devoted to the stability

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and convergence of numerical methods for VFDEs (see, e.g., [2,3,15,24,29–31,35,36,51,52,55]). In this paper, we investigate the dissipativity of numerical methods for VFDEs (1.1) and apply the results obtained to Nicholson blowflies model with variable delay or distributed delay, and diffusive Nicholson blowflies equations with multiple delays.

Many interesting problems in physics and engineering are modeled by dissipative dynamical systems which are characterized by the property of possessing a bounded absorbing set that all trajectories enter in a finite time and thereafter remain inside (see, e.g., [38,41,44]). From a computational point of view, it is important to study the potential of numerical methods in preserving the qualitative behavior of the underlying system. At the end of the last century, many papers have been focused on the dissipativity analysis of the exact and numerical solution of ordinary differential equations (ODEs, see [18,19,23,40,41]). Delay different equations (DDEs) and more general VFDEs are essentially different from ODEs because they are infinite dimensional in the sense of their phase space. Since the last decade, a very intensive investigation of numerical methods for dissipative DDEs has been started (see, for example, [13,20–22,45–47]). The numerical dissipativity of integro-differential equations (IDEs) and delay integro-differential equations (DIDEs) has been also investigated [11,12,37]. Recently, the exact and numerical dissipativity of neutral delay differential equations (NDDEs, see, [14,49,50,53,56,60,65]) and neutral delay integro-differential equations (NDIDEs, see, e.g., [34,61,62,66]) also have been investigated. The dissipative investigation of equations (1.1) is less developed than the study of their particular cases, DDEs and DIDEs. Among those papers related to our discussions on (1.1) we refer to the papers [48,52,58,59,63]. Observe that all these numerical dissipativity investigations into VFDEs including its special cases, DDEs, IDEs and DIDEs, are based on numerical approximation by fixed time-stepping methods. However, most algorithms used in practice allow the timesteps to change from one step to the next. In fact, variable stepsizes are often essential to obtain computationally efficient, accurate results for solutions of time dependent differential equation with different time scales since the variable step-size methods allow us take different time stepsizes for different time scales, i.e., small time step-sizes for the time domain with solution rapidly varying and large for the time domain with solutions slowly changing (see, e.g., recent paper [57]). For VFDEs, variable stepsizes methods are even more important because the solution to this class of equations is generally not sufficiently smooth and has breaking points (see, e.g., [2,3,25]). Then, one of the purpose of this paper is to study the numerical dissipativity of the variable-stepsize RK methods applied to the nonlinear VFDEs (1.1) because they are one of the most popular time-stepping methods (see, e.g., [8,9,17,29]).

Let us denote by $\mathbf{H}$ the real or complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and corresponding norm $\|\cdot\|$. Let $\mathbf{X}$ be a dense continuously embedded subspace of $\mathbf{H}$, and $B(0,r)\equiv\{x\in\mathbf{H}:\|x\|<r\}$ for any $r>0$. Let Z^+ denote the set of all positive integers. Define $C_{\mathbf{X}}[a,b]$ as the set of all continuous functions on interval $[a,b]$. Suppose that $f:[0,+\infty)\times\mathbf{X}\times C_{\mathbf{X}}[-\tau,+\infty)\to\mathbf{H}$ is a locally Lipschitz continuous function and satisfies the following conditions:

$$\Re e\langle f(t,y,\psi(\cdot)),y\rangle\le\gamma+\alpha\|y\|^2+\beta\max_{t-\mu_2(t)\le\xi\le t-\mu_1(t)}\|\psi(\xi)\|^2,$$
$$t\ge0, y\in\mathbf{X}, \psi\in C_{\mathbf{X}}[-\tau,+\infty), \tag{1.2}$$

where the functions $\mu_1(t)$ and $\mu_2(t)$ are assumed to satisfy

$$0\le\mu_1(t)\le\mu_2(t)\le t+\tau, \qquad \forall t\ge0, \tag{1.3}$$

$$t-\mu_2(t)\to+\infty \qquad \text{as} \qquad t\to+\infty, \tag{1.4}$$

and $\alpha<0,\ \beta\ge0,\ \gamma\ge0$ are constants. Note that the conditions (1.3) and (1.4) allow the delay to be infinite. Then the proportional delay differential equations are covered by the initial value problem (1.1).

A bounded closed set $\mathcal{B}$ is called an absorbing set for (1.1) if $\forall\phi\in C_{\mathbf{X}}[-\tau,0]$, there exists $t^*=t^*(\phi)$ such that $y(t)\in\mathcal{B}$ $\forall t\ge t^*$. The system (1.1) is termed dissipative if a bounded absorbing set exists. Then under the structural assumption (1.2), (1.1) is dissipative as stated in the following theorem [59].

Proposition 1.1 (see Wen, Yu and Wang [59]). *Suppose that $y(t)$ is a solution of the problem (1.1) satisfying the condition (1.2)–(1.4) and*

$$\alpha+\beta<0. \tag{1.5}$$

Then

(i) *for any given $\epsilon>0$, there exists a positive number $t^*=t^*(\|\phi\|,\epsilon)$, such that*

$$\|y(t)\|^2\le\frac{\gamma}{-(\alpha+\beta)}+\epsilon, \qquad \forall t\ge t^*; \tag{1.6}$$

(ii) *for any given $\epsilon>0$, the system (1.1) is dissipative with an absorbing set $B=B(0,\sqrt{\gamma/[-(\alpha+\beta)]+\epsilon})$.*

The problem whether the numerical and exact solutions admit a related asymptotic behavior on the unbounded domain is an important theoretical question about their numerical approximations. On the basis of the above analytical dissipativity results, Wen et al. [63] and Wang [52] investigated simultaneously the numerical dissipativity of (k,l)-algebraically stable RK methods. It follows from the performed analysis in Wen et al. [63] that if

$$h(\alpha+p\beta)<2l, \tag{1.7}$$

where h is a fixed time stepsize and

$$p = \frac{dc_\pi^2}{C_1 - C_2} \geq 1 \tag{1.8}$$

(the definitions of d, c_π, C_1 and C_2 are given in Sections 2 and 3), then a (k, l)-algebraically stable RK method with $k \leq 1$ for nonlinear VFDEs (1.1) is dissipative. Comparing the conditions (1.5) and (1.7) for the numerical and exact dissipativity, We can find a gap between these conditions. On the one hand, it is easy to verify that a (k, l)-algebraically stable RK method with $k \leq 1$ has $l \leq 0$, which implies that

$$\alpha + p\beta < 0. \tag{1.9}$$

Condition (1.9) with $p > 1$ is obviously stronger than the exact dissipative condition (1.5). Comparison of conditions (1.9) and (1.5) also reveals that the methods with $p = 1$ can retain the dissipative property of the underlying system for all stepsize $h > 0$. For this type of methods, from the result in Wen et al. [63] we can only find one, the implicit Euler method with linear interpolation. Thus, searching other methods with $p = 1$ motivate us to study further the dissipativity of the high order RK methods. On the other hand, when condition (1.5) holds, which implies the system (1.1) is dissipative, but for a (k, l)-algebraically stable RK method with $p > 1$, if

$$\alpha + p\beta \geq 0, \tag{1.10}$$

we can not obtain any results for this method from the paper [63]. The aim of this paper is, among others, to fill in this gap and investigate the conditional dissipativity of numerical methods, that is, under a stepsize restriction, the method can preserve exactly the dissipativity of the underlying system. Note that the conditional stability of RK methods for variable delay DDEs has been investigated in Wang and Li [51].

Nicholson blowflies model and its formulations using discrete, periodic, and diffusive equations have been extensively studied in the literature; see [4,6,16,32,33,39,42,43] and references therein. In this paper, we will apply our theoretical results and dissipativity-preserving numerical methods to nonlinear Nicholson blowflies, including variable delay model, integro-differential Nicholson's model, and diffusive model with multiple delays.

The paper is organized as follows. In Section 2, for the presentation of the subsequent results, we introduce the variable-stepsize RK methods applied to VFDEs and review some related concepts and conclusions. In Section 3, the conditional dissipativity of variable-stepsize RK methods is analyzed and hence some numerical dissipative criteria are derived. In Section 4, in order to relax the restriction of condition (1.9) and reduce the value of parameter p, a sharper approach to compute C_1 and C_2 is introduced. In Section 5, we further study the dissipativity of algebraically stable RK methods and obtain a sharper numerically dissipative result. In Section 6, for giving an numerical illustration to our dissipative results, we apply some concrete variable-stepsize RK methods to the three classes of nonlinear Nicholson's blowflies models. The presented numerical examples verify our theoretical results. In Section 7, we summary the whole paper and compare the existed correlative results and our findings.

2. The variable-stepsize RK methods applied to VFDEs

In this section, we will consider the application of variable-stepsize RK methods to (1.1). Let (A, b^T, c) denote a given RK method characterized by the $s \times s$ matrix $A = (a_{ij})$ and vectors $b = [b_1, \ldots, b_s]^T$, $c = [c_1, \ldots, c_s]^T$. In this paper we always assume that $\sum_{j=1}^s b_j = 1$ and $c_i = \sum_{j=1}^s a_{ij} \in [0, 1]$. An s-stage RK method (A, b^T, c) for ODEs together with an appropriate piecewise interpolation operator π^h can generally leads to an s-stage RK method (A, b^T, c, π^h):

$$\begin{cases} y^h(t) = \pi^h(t; \phi, y_1, y_2, \ldots, y_{n+1}), & -\tau \leq t \leq t_{n+1}, \\ Y_i^{(n)} = y_n + h_{n+1} \sum_{j=1}^s a_{ij} f(t_{n,j}, Y_j^{(n)}, y^h(\cdot)), & i = 1, \ldots, s, \\ y_{n+1} = y_n + h_{n+1} \sum_{j=1}^s b_j f(t_{n,j}, Y_j^{(n)}, y^h(\cdot)), \end{cases} \tag{2.1}$$

for solving problem (1.1) in VFDEs. Here, t_n are mesh points, $h_{n+1} = t_{n+1} - t_n$, $t_{n,j} = t_n + c_j h_{n+1}$, the interpolation function $y^h(t)$ is an approximation to the true solution $y(t)$ of the problem (1.1), y_n and $Y_j^{(n)}$ are approximations to $y(t_n)$ and $y(t_{n,j})$, respectively. For simplicity, we always assume that the interpolation operator π^h satisfies the following condition:

$$\max_{t_* \leq t \leq t_n} \|\pi^h(t; \phi, y_1, \ldots, y_n)\| \leq \begin{cases} c_\pi \max_{\eta(t_*) \leq i \leq n} \|y_i\|, & \eta(t_*) \geq 0, \\ c_\pi \max\{\max_{1 \leq i \leq n} \|y_i\|, \max_{-\tau \leq t \leq 0} \|\phi(t)\|\}, & \eta(t_*) < 0, \end{cases} \tag{2.2}$$

where $-\tau \leq t_* \leq t_n$, $y_i \in \mathbf{X}$, $\quad i = 1, 2, \ldots, n$. The integer n is assumed to be greater than $\mathcal{N}$, where $\mathcal{N}$ denotes the number of additional starting values, which are occasionally needed for the interpolation at the first integration step. The function $\eta(t)$ is defined by

$$\eta(t) = \min\left\{m : \ m \in Z^+, \ t_m \geq t\right\} - p_*, \tag{2.3}$$

where p_* denotes a positive integer depending only on the procedure of the interpolation. The constant $c_\pi \geq 1$ is of moderate size and independent of t_*, n, y_i and ϕ. In this paper we consider quasi-uniform meshes, that is, there exists a constant $c_* \geq 1$ such that $1/c_* \leq h_n/h_{n+1} \leq c_*$. Let $h_{\min} = \inf_{n \in Z^+} h_n$ and $h_{\max} = \sup_{n \in Z^+} h_n$.

We will assume throughout the paper that for every implicit Eq. (2.1) there exists a unique solution $Y^{(n)} = [Y_1^{(n)}, \dots, Y_s^{(n)}] \in \mathbf{X}^s$. For the sake of brevity, we introduce the following notations. For any nonnegative diagonal matrix $D = \operatorname{diag}(d_1, d_2, \dots, d_s)$, we define a pseudo-inner product and the corresponding pseudo-norm on $\mathbf{X}^s$ by

$$\langle Y, Z \rangle_D = \sum_{j=1}^{s} d_j \langle Y_j, Z_j \rangle, \quad \|Y\|_D = \langle Y, Y \rangle_D^{1/2},$$
$$Y = [Y_1, Y_2, \dots, Y_s] \in \mathbf{X}^s, \quad Z = [Z_1, Z_2, \dots, Z_s] \in \mathbf{X}^s.$$

It is easy to verify that when D is positive definite, they are the inner product and the norm on $\mathbf{X}^s$ respectively.

The following concepts for RK method (A, b^T, c) will play the important roles in the subsequent analysis.

Definition 2.1. (see Burrage and Butcher [5]) Let k, l be real constants. A RK method (A, b^T, c) (or (A, b^T, c, π^h)) is said to be (k, l)-algebraically stable if there exists a diagonal nonnegative matrix $D = \operatorname{diag}(d_1, d_2, \dots, d_s)$ such that $\mathcal{M} = [\mathcal{M}_{ij}]$ is nonnegative definite, where

$$\mathcal{M} = \begin{bmatrix} k - 1 - 2le^T De & e^T D - b^T - 2le^T DA \\ De - b - 2lA^T De & DA + A^T D - bb^T - 2lA^T DA \end{bmatrix}, \tag{2.4}$$

$A = [a_{ij}] \in R^{s \times s}$, $b = [b_1, b_2, \dots, b_s]^T \in R^s$ and $e = [1, 1, \dots, 1]^T$. In particular, a (1,0)-algebraically stable method is called algebraically stable.

Some (k, l)-algebraically stable RK methods have been given in Burrage and Butcher [5] and Hairer and Wanner [17]. Generally, given l, we can calculate the value k. Since D is a nonnegative matrix, it is easy to verify that $l \le 0$ if $k \le 1$.

Definition 2.2. The stability function of a RK method (A, b^T, c) is defined by

$$R(z) = 1 + zb^T(I - zA)e. \tag{2.5}$$

Definition 2.3. (See [17]). The method (A, b^T, c) is said to be strongly stable at ∞ if

$$|R(\infty)| = |1 - b^T A^{-1} e| < 1, \tag{2.6}$$

and said to be stiffly accurate if

$$a_{si} = b_i, \qquad i = 1, 2, \dots, s. \tag{2.7}$$

Definition 2.4. (See [7,9]). A method is *DJ*-reducible if, for some non-empty index set $S \subset \{1, \dots, s\}$,

$$b_j = 0 \quad \text{for} \quad j \in S \qquad \text{and} \qquad a_{ij} = 0 \quad \text{for} \quad i \notin S, \ j \in S. \tag{2.8}$$

It is otherwise *DJ*-irreducible.

We recall from Dahlquist and Jeltsch [7] (see also [9]), that a *DJ*-irreducible, algebraically stable method satisfies

$$b_i > 0, \qquad i = 1, 2, \dots, s. \tag{2.9}$$

Definition 2.5. (see [20,23,41]). A RK method (A, b^T, c, π^h) is said to be dissipative if, whenever the method is applied with variable stepsize h_n to a dynamical system of the form (1.1) subject to (1.2)–(1.5), there exists a constant r such that, for any function $\phi(t)$, there exists an $N_0(\bar{\phi}, h_{\min}, h_{\max})$, $\bar{\phi} = \max\left\{\sup_{-\tau \le t \le 0} \|\phi(t)\|, \ \max_{1 \le i \le \mathcal{N}} \|y_i\|\right\}$, such that

$$\|y_n\| \le r, \quad n \ge N_0. \tag{2.10}$$

3. The dissipativity of variable-stepsize RK methods

This section will focus on the dissipative analysis of variable-stepsize RK methods for system (1.1) and establish the dissipative criteria of the methods. To this end, we first give a lemma.

Lemma 3.1. *Let problem (1.1) satisfy the condition (1.2)–(1.4). Assume that the method (A, b^T, c, π^h) for (1.1) is (k, l)-algebraically stable with nonnegative matrix $D = \operatorname{diag}(d_1, d_2, \dots, d_s)$ and $0 \le k \le 1$, the interpolation operator π^h satisfies condition (2.2), and there exist constants $C_1 > C_2 \ge 0$, which depend only on the method, such that*

$$\Re e\langle Y^{(n)}, DY^{(n)} \rangle \ge C_1 \|y_{n+1}\|^2 - C_2 \|y_n\|^2 \tag{3.1}$$

and

$$2(\alpha h_{n+1} - l)(C_1 - C_2) + 2c_\pi^2 \beta d h_{n+1} \le 1 - k. \tag{3.2}$$

Then, when $\alpha h_{n+1} \le l$, we have

$$\|y_{n+1}\|^2 \le \begin{cases} \sigma(h_{n+1}) \max_{\hat{n} \le i \le n} \|y_i\|^2 + r_1(h_{n+1}), & \text{for } \eta(\hat{t}_n) \ge 0, \\ \sigma(h_{n+1}) \max\left\{\max_{1 \le i \le n} \|y_i\|^2, \bar{\phi}^2\right\} + r_1(h_{n+1}), & \text{for } \eta(\hat{t}_n) < 0, \end{cases} \tag{3.3}$$

where the functions $\sigma(h)$ and $r_1(h)$ are defined by

$$\sigma(h) = \frac{k - 2(\alpha h - l)C_2 + 2c_\pi^2 \beta dh}{1 - 2(\alpha h - l)C_1}, \quad r_1(h) = \frac{2hd\gamma}{1 - 2(\alpha h - l)C_1 - 2c_\pi^2 \beta dh},$$

and

$$d = \sum_{j=1}^{s} d_j, \qquad \hat{t}_n = \min_{1 \le i \le s} \{t_n + c_i h_{n+1} - \mu_2(t_n + c_i h_{n+1})\}, \qquad \hat{n} = \min\left\{n, \eta(\hat{t}_n)\right\}.$$

Proof. As in Burrage and Butcher [5] and Humphries and Stuart [23], we can easily obtain that

$$\|y_{n+1}\|^2 - k\|y_n\|^2 - 2\sum_{j=1}^{s} d_j \Re e\langle Y_j^{(n)}, h_{n+1} f(t_{n,j}, Y_j^{(n)}, y^h(\cdot)) - lY_j^{(n)}\rangle = -\sum_{i=1}^{s}\sum_{j=1}^{s} \mathcal{M}_{ij}\langle Q_i, Q_j\rangle, \tag{3.4}$$

where $Q_1 = y_n$, $Q_i = h_{n+1} f(t_{n,i-1}, Y_{i-1}^{(n)}, y^h(\cdot))$, $i = 2, 3, \ldots, s+1$.

Using the (k, l)-algebraic stability of the method and (1.2), we have

$$\|y_{n+1}\|^2 \le k\|y_n\|^2 + 2h_{n+1} d\gamma + 2(\alpha h_{n+1} - l)\|Y^{(n)}\|_D^2 + 2\beta d h_{n+1} \max_{\hat{t}_n \le t \le t_{n+1}} \|y^h(t)\|^2. \tag{3.5}$$

By the condition (3.1) and $\alpha h_{n+1} \le l$, from the above inequality, we further give

$$\|y_{n+1}\|^2 \le k\|y_n\|^2 + 2h_{n+1} d\gamma + 2(\alpha h_{n+1} - l)\left[C_1\|y_{n+1}\|^2 - C_2\|y_n\|^2\right] + 2\beta d h_{n+1} \max_{\hat{t}_n \le t \le t_{n+1}} \|y^h(t)\|^2,$$

which leads to

$$[1 - 2(\alpha h_{n+1} - l)C_1]\|y_{n+1}\|^2 \le [k - 2(\alpha h_{n+1} - l)C_2]\|y_n\|^2 + 2h_{n+1} d\gamma + 2\beta d h_{n+1} \max_{\hat{t}_n \le t \le t_{n+1}} \|y^h(t)\|^2. \tag{3.6}$$

By means of the canonical condition (2.2), we get

$$\begin{aligned} \max_{\hat{t}_n \le t \le t_{n+1}} \|y^h(t)\|^2 &= \max_{\hat{t}_n \le t \le t_{n+1}} \|\pi^h(t; \varphi, y_1, \ldots, y_{n+1})\|^2 \\ &\le \begin{cases} c_\pi^2 \max_{\eta(\hat{t}_n) \le i \le n+1} \|y_i\|^2, & \eta(\hat{t}_n) \ge 0, \\ c_\pi^2 \max\left\{\max_{1 \le i \le n+1} \|y_i\|^2, \max_{-\tau \le t \le 0} \|\phi(t)\|^2\right\}, & \eta(\hat{t}_n) < 0. \end{cases} \end{aligned} \tag{3.7}$$

Now we consider the following two cases successively.

Case 1. $\eta(\hat{t}_n) \ge 0$. In this case, substitute (3.7) into (3.6) to obtain

$$\begin{aligned} [1 - 2(\alpha h_{n+1} - l)C_1]\|y_{n+1}\|^2 &\le [k - 2(\alpha h_{n+1} - l)C_2]\|y_n\|^2 + 2h_{n+1} d\gamma \\ &\quad + 2\beta d h_{n+1} c_\pi^2 \max_{\eta(\hat{t}_n) \le i \le n+1} \|y_i\|^2. \end{aligned} \tag{3.8}$$

If $\max_{\eta(\hat{t}_n) \le i \le n+1} \|y_i\|^2 = \|y_{n+1}\|^2$, noting that condition (3.2) implies $1 - 2(\alpha h_{n+1} - l)C_1 - 2\beta c_\pi^2 d h_{n+1} > 0$ and

$$\frac{k - 2(\alpha h_{n+1} - l)C_2}{1 - 2(\alpha h_{n+1} - l)C_1 - 2c_\pi^2 \beta d h_{n+1}} \le \frac{k - 2(\alpha h_{n+1} - l)C_2 + 2c_\pi^2 \beta d h_{n+1}}{1 - 2(\alpha h_{n+1} - l)C_1},$$

then we have

$$\begin{aligned} \|y_{n+1}\|^2 &\le \frac{k - 2(\alpha h_{n+1} - l)C_2}{1 - 2(\alpha h_{n+1} - l)C_1 - 2c_\pi^2 \beta d h_{n+1}}\|y_n\|^2 + \frac{2h_{n+1} d\gamma}{1 - 2(\alpha h_{n+1} - l)C_1 - 2\beta c_\pi^2 d h_{n+1}} \\ &\le \sigma(h_{n+1})\|y_n\|^2 + r_1(h_{n+1}); \end{aligned} \tag{3.9}$$

otherwise, (3.8) yields

$$\|y_{n+1}\|^2 \le \frac{k - 2(\alpha h_{n+1} - l)C_2 + 2c_\pi^2 \beta d h_{n+1}}{1 - 2(\alpha h_{n+1} - l)C_1} \max_{\eta(\hat{t}_n) \le i \le n} \|y_i\|^2 + \frac{2h_{n+1} d\gamma}{1 - 2(\alpha h_{n+1} - l)C_1}. \tag{3.10}$$

Case 2. $\eta(\hat{t}_n) \le 0$. It follows from (3.6) and (3.7) that

$$\begin{aligned} [1 - 2(\alpha h_{n+1} - l)C_1]\|y_{n+1}\|^2 &\le [k - 2(\alpha h_{n+1} - l)C_2]\|y_n\|^2 + 2h_{n+1} d\gamma \\ &\quad + 2\beta d h_{n+1} c_\pi^2 \max\left\{\max_{1 \le i \le n} \|y_i\|^2, \max_{-\tau \le t \le 0} \|\phi(t)\|^2\right\}. \end{aligned} \tag{3.11}$$

Then, following a similar line, (3.11) yields (3.3).

Thus, the proof of Lemma 3.1 is completed. □

Notice that for the case of $\beta = 0$, we get the following inequality

$$\|y_{n+1}\|^2 \le \frac{k - 2(\alpha h_{n+1} - l)C_2}{1 - 2(\alpha h_{n+1} - l)C_1}\|y_n\|^2 + \frac{2h_{n+1} d\gamma}{1 - 2(\alpha h_{n+1} - l)C_1}, \quad \alpha h_{n+1} \le l. \tag{3.12}$$

Based on Lemma 3.1, we give the main result of this section.

Theorem 3.1. *Let problem (1.1) satisfy condition (1.2)–(1.4). Assume that the method* (A, b^T, c, π^h) *for (1.1) is* (k, l)*-algebraically stable with nonnegative matrix* $D = \mathrm{diag}(d_1, d_2, \ldots, d_s)$ *and* $0 \le k \le 1$*, the interpolation operator* π^h *satisfies condition (2.2), and there exist constants* $C_1 > C_2 \ge 0$*, which depend only on the method, such that (3.1) holds. Then the method is dissipative whenever*

$$\alpha h_{n+1} \le l, \quad (\alpha + p\beta)h_{n+1} < q, \quad \forall n \ge 1, \tag{3.13}$$

where

$$p = \frac{dc_\pi^2}{C_1 - C_2}, \qquad q = l + \frac{1-k}{2(C_1 - C_2)}.$$

Proof. As done in Li [29], we can construct a strictly increased sequence of integers $\{n_\varpi\}$ which diverges to $+\infty$ as $\varpi \to +\infty$, such that

$$t - \tau_2(t) > t_{n_\varpi + p_*}, \quad \forall t > t_{n_{\varpi+1}},$$

where $n_0 = 0$. In fact, suppose that n_ϖ $(\varpi \ge 0)$ has been chosen appropriately. Since $\lim_{t\to+\infty}(t - \tau_2(t)) = +\infty$, we can get a constant $M_1 > 0$ such that for all $t > M_1$, the inequality $t - \tau_2(t) > t_{n_\varpi + p_*}$ holds. Since $\lim_{n\to+\infty} t_n = +\infty$, we can get a constant $M_2 > 0$ such that for all $n \ge M_2$, the inequality $t_n > M_1$ holds. We thus choose $n_{\varpi+1} = \max\{n_\varpi + 1, M_2, \mathcal{N}\}$, in order to have the relations $n_{\varpi+1} > n_\varpi$ and

$$\hat{n} =: \min\left\{n, \eta(\hat{t}_n)\right\} > n_\varpi, \qquad n \ge n_{\varpi+1}. \tag{3.14}$$

Observe first that condition $(\alpha + p\beta)h_{n+1} < q$ implies that (3.2) holds. Then for any given integers $\varpi \ge 1$ and $j = 1, 2, \ldots, n_{\varpi+1} - n_\varpi$, it follows from (3.3) that

$$\begin{aligned} \|y_{n_\varpi+j}\|^2 &\le \Gamma(h_{\min}, h_{\max}) \max_{\widehat{n_\varpi+j-1} \le i \le n_\varpi+j-1} \|y_i\|^2 + \Phi(h_{\min}, h_{\max}) \\ &\le \Gamma(h_{\min}, h_{\max}) \max_{n_{\varpi-1}+1 \le i \le n_\varpi+j-1} \|y_i\|^2 + \Phi(h_{\min}, h_{\max}), \end{aligned} \tag{3.15}$$

where

$$\Gamma(h_{\min}, h_{\max}) = \begin{cases} \sigma(h_{\min}) & \text{if } \alpha(C_1k - C_2) + c_\pi^2\beta d(1 + 2lC_1) < 0, \\ \sigma(h_{\max}) & \text{if } \alpha(C_1k - C_2) + c_\pi^2\beta d(1 + 2lC_1) \ge 0; \end{cases} \tag{3.16}$$

$$\Phi(h_{\min}, h_{\max}) = \begin{cases} r_1(h_{\min}) & \text{if } 1 + 2lC_1 < 0, \\ r_1(h_{\max}) & \text{if } 1 + 2lC_1 \ge 0. \end{cases} \tag{3.17}$$

Then we further have

$$\max_{n_\varpi < i \le n_{\varpi+1}} \|y_i\|^2 \le \Gamma(h_{\min}, h_{\max}) \max_{n_{\varpi-1} < i \le n_{\varpi+1}} \|y_i\|^2 + \Phi(h_{\min}, h_{\max}), \tag{3.18}$$

where we have used the fact that when $\alpha(C_1k - C_2) + c_\pi^2\beta d(1 + 2lC_1) < 0$ the function $\sigma(h)$ is monotone decreasing, and conversely it is monotone increasing. The similar property of the function $r_1(h)$ is also used here.

As an important step toward the proof of this theorem, we show that

$$\max_{n_\varpi < i \le n_{\varpi+1}} \|y_i\|^2 \le \Gamma(h_{\min}, h_{\max}) \max_{n_{\varpi-1} < i \le n_\varpi} \|y_i\|^2 + \frac{\Phi(h_{\min}, h_{\max})}{1 - \Gamma(h_{\min}, h_{\max})}, \tag{3.19}$$

where we have used $\Gamma(h_{\min}, h_{\max}) < 1$ which can be derived from $(\alpha + p\beta)h_n < q$, $n \in Z^+$.

In fact, if $\max_{n_{\varpi-1} < i \le n_{\varpi+1}} \|y_i\|^2 = \max_{n_\varpi < i \le n_{\varpi+1}} \|y_i\|^2$, from (3.18), one easily gets

$$\max_{n_\varpi < i \le n_{\varpi+1}} \|y_i\|^2 \le \Gamma(h_{\min}, h_{\max}) \max_{n_\varpi < i \le n_{\varpi+1}} \|y_i\|^2 + \Phi(h_{\min}, h_{\max}),$$

which leads to

$$\max_{n_\varpi < i \le n_{\varpi+1}} \|y_i\|^2 \le \frac{\Phi(h_{\min}, h_{\max})}{1 - \Gamma(h_{\min}, h_{\max})}; \tag{3.20}$$

otherwise, (3.18) yields

$$\max_{n_\varpi < i \le n_{\varpi+1}} \|y_i\|^2 \le \Gamma(h_{\min}, h_{\max}) \max_{n_{\varpi-1} < i \le n_\varpi} \|y_i\|^2 + \Phi(h_{\min}, h_{\max}). \tag{3.21}$$

Now, from (3.19), by induction, we have

$$\max_{n_\varpi < i \le n_{\varpi+1}} \|y_i\|^2 \le (\Gamma(h_{\min}, h_{\max}))^j \max_{n_{\varpi-j} < i \le n_{\varpi-j+1}} \|y_i\|^2 + \frac{\Phi(h_{\min}, h_{\max})}{1 - \Gamma(h_{\min}, h_{\max})} \sum_{j=0}^{j-1} (\Gamma(h_{\min}, h_{\max}))^j. \tag{3.22}$$

Then for any $\epsilon > 0$, letting

$$r(h_{\min}, h_{\max}) = \frac{\sqrt{\Phi(h_{\min}, h_{\max})}}{1 - \Gamma(h_{\min}, h_{\max})} + \epsilon,$$

and using (3.22), we have that there exists an N_0, which depends on $\phi(t)$, additional starting values $y_1, \dots, y_{\mathcal{N}}$, $h_{\min}$, and $h_{\max}$, such that

$$\|y_n\| \le r(h_{\min}, h_{\max}), \quad n \ge N_0.$$

This completes the proof. □

If we go into the details of the induction, we can obtain the following inequality

$$\|y_n\| \le r(h_{\min}, h_{\max}) = \sqrt{\frac{\Phi(h_{\min}, h_{\max})}{1 - \Gamma(h_{\min}, h_{\max})}} + \epsilon, \quad n \ge N_0. \tag{3.23}$$

As a matter of fact, if there exists an integer $\varpi \ge 1$ such that (3.20) holds, then for $\max_{n_{\varpi+1} < i \le n_{\varpi+2}} \|y_i\|^2$, we have, from (3.20)

$$\max_{n_{\varpi+1} < i \le n_{\varpi+2}} \|y_i\|^2 \le \frac{\Phi(h_{\min}, h_{\max})}{1 - \Gamma(h_{\min}, h_{\max})};$$

or, from (3.21),

$$\begin{aligned} \max_{n_{\varpi+1} < i \le n_{\varpi+2}} \|y_i\|^2 &\le \Gamma(h_{\min}, h_{\max}) \max_{n_{\varpi} < i \le n_{\varpi+1}} \|y_i\|^2 + \Phi(h_{\min}, h_{\max}) \\ &\le \Gamma(h_{\min}, h_{\max}) \frac{\Phi(h_{\min}, h_{\max})}{1 - \Gamma(h_{\min}, h_{\max})} + \Phi(h_{\min}, h_{\max}) \\ &= \frac{\Phi(h_{\min}, h_{\max})}{1 - \Gamma(h_{\min}, h_{\max})}. \end{aligned} \tag{3.24}$$

Then by induction, we get (3.23). In the other case where for any integer $\varpi \ge 1$, (3.20) doesn't hold, we get (3.23) from (3.21).

In the practical usage of Theorem 3.1, for a $(k, l)-$algebraically stable RK method, we can consider the following two cases:

Case 1. The condition (1.9) holds, that is, $\alpha + p\beta < 0$. Then when stepsize h_{n+1} satisfies

$$\max\left\{\frac{l}{\alpha}, \frac{q}{\alpha + p\beta}\right\} \le h_{n+1},$$

this method is dissipative; especially, for an algebraically stable RK method, we have the following corollary which states that for any variable stepsize this method is dissipative.

Corollary 3.1. *Let problem (1.1) satisfy condition (1.2)–(1.4). Assume that the method (A, b^T, c, π^h) for (1.1) is algebraically stable, the interpolation operator π^h satisfies condition (2.2), and there exist constants $C_1 > C_2 \ge 0$, which depend only on the method, such that (3.1) holds. Then, for any variable stepsize, the method (A, b^T, c, π^h) is dissipative whenever $\alpha + p\beta < 0$ with*

$$p = \frac{c_\pi^2}{C_1 - C_2}.$$

Proof. Noting that in this case, $D = B = \mathrm{diag}(b_1, b_2, \dots, b_s)$, $\alpha(C_1 k - C_2) + c_\pi^2 \beta d(1 + 2lC_1) = \alpha(C_1 - C_2) + c_\pi^2 \beta d < 0$ and $\sigma(h)$ is a decreasing function, as done in Theorem 3.1, we can easily prove this corollary. □

Case 2. The condition (1.10) holds, that is, $\alpha + p\beta \ge 0$, and $\alpha + \beta < 0$, which implies that the underlying system is dissipative. Let us emphasize that for this case we can not obtain any results from previous literatures. But from Theorem 3.1 when $q > 0$ we can choose the stepsize h_{n+1} satisfying

$$\frac{l}{\alpha} \le h_{n+1} < \frac{q}{\alpha + p\beta},$$

such that this method is dissipative.

Now let us give some examples to illustrate these cases.

Example 3.1. Consider one-leg θ-method

$$\begin{array}{c|c} \theta & \theta \\ \hline & 1 \end{array}$$

where we assume $\theta \ge 1/2$. From [5], we know that when $l < 1/\theta$ it is (k, l)-algebraically stable with $k = \chi^2$ and

$$\chi = \begin{cases} \frac{1 + l(1-\theta)}{1 - l\theta}, & l \ge -\frac{2\theta - 1}{2\theta(1-\theta)}, \\ \frac{1-\theta}{\theta}, & l \le -\frac{2\theta - 1}{2\theta(1-\theta)}. \end{cases}$$

On the other hand, we have

$$\|Y^{(n)}\|^2 = \|\theta y_{n+1} + (1-\theta)y_n\|^2 \geq (2\theta-1)[\theta\|y_{n+1}\|^2 - (1-\theta)\|y_n\|^2], \tag{3.25}$$

which implies that the method satisfies condition (3.1) with $C_1 = (2\theta-1)\theta\chi$ and $C_2 = (2\theta-1)(1-\theta)\chi$. Then when $\alpha + p\beta = \alpha + \frac{\beta}{(2\theta-1)^2} < 0$, we can choose $l = 0$ such that for any variable stepsize, this method with piecewise linear interpolation is dissipative. In particular, for the backward Euler method, $p = 1$, which implies that the backward Euler method with piecewise linear interpolation can preserve the dissipativity of the underlying systems. When $p = \frac{1}{(2\theta-1)^2} > 1$ and $\beta < -\alpha \leq p\beta$, we can choose $l = -\frac{2\theta-1}{2\theta(1-\theta)}$ and stepsizes satisfying

$$\frac{2\theta-1}{2\alpha\theta(1-\theta)} \leq h_{\min} \leq h_{\max} < \frac{2(2\theta-1)}{(2\theta-1)^2\alpha+\beta} \tag{3.26}$$

such that (3.13) holds, and therefore this method is dissipative under this stepsize restriction.

4. The computation of constants C_1 and C_2

It is remarkable that the inequality (3.1) and the constants C_1 and C_2 in it play the important role in the above analysis. In view of this, the author in Li [29] presented an approach to compute them for two classes of algebraically stable RK methods: (i) A nonsingular, $b > 0$, and strongly stable at ∞; (ii) $b > 0$ and stiffly accurate. In order to compute C_1 and C_2 more accurately, in the following, we give a theorem for strongly stable RK methods.

Theorem 4.1. *Assume that RK method (A, b^T, c, π^h) is DJ-irreducible and strongly stable at ∞. Then, for matrix $D = \mathrm{diag}(d_1, d_2, \dots, d_s)$ with $d_i > 0$ $(i = 1, 2, \dots, s)$, the inequality (3.1) holds with*

$$C_1 = \lambda(1-\delta), \qquad C_2 = \lambda(1-\delta)\delta, \tag{4.1}$$

where

$$\lambda = \left(\sum_{j=1}^{s} d_j^{-1}u_j^2\right)^{-1}, \quad [u_1, \dots, u_s] = \lim_{\epsilon\to 0} b^T(A+\epsilon I)^{-1}, \quad \delta = |R(\infty)| < 1.$$

Proof. Let us first suppose that A is nonsingular and denote the elements of its inverse by ω_{ij}. From the second equality of (2.1) we get

$$Q_{i+1} = \sum_{j=1}^{s}\omega_{ij}Y_j^{(n)} - \left(\sum_{j=1}^{s}\omega_{ij}\right)y_n, \quad i = 1, 2, \dots, s. \tag{4.2}$$

Insert them into the third equality of (2.1) to yield

$$y_{n+1} = R(\infty)y_n + \sum_{j=1}^{s}\left(\sum_{i=1}^{s}b_i\omega_{ij}\right)Y_j^{(n)}. \tag{4.3}$$

Then by using the Cauchy-Schwarz inequality we get

$$\begin{aligned}\left[\sum_{j=1}^{s}d_j^{-1}\left(\sum_{i=1}^{s}b_i\omega_{ij}\right)^2\right]^{1/2}\left[\sum_{j=1}^{s}d_j\|Y_j^{(n)}\|^2\right]^{1/2} &\geq \|y_{n+1} - R(\infty)y_n\| \\ &\geq \big|\|y_{n+1}\| - \delta\|y_n\|\big|,\end{aligned} \tag{4.4}$$

and consequently we have

$$\sum_{j=1}^{s}d_j\|Y_j^{(n)}\|^2 \geq \lambda(\|y_{n+1}\| - \delta\|y_n\|)^2 \geq \lambda(1-\delta)\left(\|y_{n+1}\|^2 - \delta\|y_n\|^2\right). \tag{4.5}$$

If the RK matrix A is singular, we replace it everywhere by the regular matrix $(A + \epsilon I)$ and consider the limit $\epsilon \to 0$. This completes the proof. □

If an algebraically stable RK method with A nonsingular and $b > 0$ is strongly stable at ∞, Li in Li [29] obtained $C_1 = (1-\delta)\left(\min_{1\leq j\leq s} b_j\right)\|b^TA^{-1}\|_0^{-2}$ and $C_2 = \delta C_1$. Now since $D = \mathrm{diag}(b_1, b_2, \dots, b_s)$ and $\lambda \geq \left(\min_{1\leq j\leq s} b_j\right)\|b^TA^{-1}\|_0^{-2}$, the values of constants C_1 and C_2 computed in this paper are better than those obtained in [29]. The following example demonstrates this conclusion.

Example 4.1. Consider 2-stage Radau IA method

$$\begin{array}{c|cc} 0 & 1/4 & -1/4 \\ 2/3 & 1/4 & 5/12 \\ \hline & 1/4 & 3/4 \end{array}. \tag{4.6}$$

Firstly, we have $\delta = 0$ (see also [29]). It is also easy to obtain $[u_1, \ldots, u_s] = b^T A^{-1} = [-\frac{1}{2}, \frac{3}{2}]$. Then when we take $D = \text{diag}(1/4, 3/4)$, we get $\lambda = 1/4$ and therefor $C_1 = 1/4$ and $C_2 = 0$. Since it is well-known that 2-stage Radau IA method (4.6) is algebraically stable with $D = \text{diag}(1/4, 3/4)$, the computed values $C_1 = 1/4$ and $C_2 = 0$ improve the previous results obtained in Li [29] with $C_1 = 1/10$ and $C_2 = 0$.

5. A further analysis to the algebraically stable RK methods

Comparing condition (1.5) and (1.9), we find that there exists a gap between them when $p > 1$. In this section, we further consider algebraically stable RK methods, where we will take use of the correlative approach in Li [29] and the previous idea that relaxing condition (1.9) by a smaller value of constant p. To do this, we make the following assumption:

$\mathcal{A}1$. There exist constants $C_1 > C_2 \geq 0$ and $\Gamma \geq 0$, and nonnegative functions $\varphi_1(h)$, $\varphi_2(h)$, and $\varphi_3(h) \leq \Gamma$, such that

$$\Re e\langle Y^{(n)}, BY^{(n)}\rangle \geq [C_1 + \varphi_2(h_{n+1})]\varphi_1(h_{n+1})\|y_{n+1}\|^2 - [C_2 + \varphi_2(h_{n+1})]\varphi_1(h_{n+1})\|y_n\|^2 - \left[\frac{(\varphi_1(h_{n+1}) - 1)\beta}{-\alpha}\right] \max_{\hat{t}_n \leq t \leq t_{n+1}} \|y^h(t)\|^2 - \varphi_3(h_{n+1}). \tag{5.1}$$

where the constants C_1, C_2, Γ depend only on the method, the functions $\varphi_1(h)$, $\varphi_2(h)$, and $\varphi_3(h)$ depend only on α, β, γ, and the method.

Then we have the following theorem.

Theorem 5.1. *Let problem (1.1) satisfy conditions (1.2)–(1.4). Assume that the method (A, b^T, c, π^h) for (1.1) is algebraically stable and satisfies the assumption $\mathcal{A}1$, the interpolation operator π^h satisfies condition (2.2). Then, for any variable stepsize, the method (A, b^T, c, π^h) is dissipative whenever $\alpha + p\beta < 0$ with*

$$p = \frac{c_\pi^2}{C_1 - C_2}.$$

Proof. Similar to Theorem 3.1, from the algebraical stability of the method, the conditions (1.2)-(1.4) and (5.1), we have

$$\begin{aligned}
\|y_{n+1}\|^2 \leq\ & \|y_n\|^2 + 2\alpha h_{n+1}[C_1 + \varphi_2(h_{n+1})]\varphi_1(h_{n+1})\|y_{n+1}\|^2 \\
& -2\alpha h_{n+1}[C_2 + \varphi_2(h_{n+1})]\varphi_1(h_{n+1})\|y_n\|^2 \\
& -2\alpha h_{n+1}\left[\frac{(\varphi_1(h_{n+1}) - 1)\beta}{-\alpha}\right] \max_{\hat{t}_n \leq t \leq t_{n+1}} \|y^h(t)\|^2 \\
& -2\alpha h_{n+1}\varphi_3(h_{n+1}) + 2h_{n+1}\gamma + 2\beta h_{n+1} \max_{\hat{t}_n \leq t \leq t_{n+1}} \|y^h(t)\|^2 \\
\leq\ & 2\alpha h_{n+1}[C_1 + \varphi_2(h_{n+1})]\varphi_1(h_{n+1})\|y_{n+1}\|^2 \\
& +\{1 - 2\alpha h_{n+1}[C_2 + \varphi_2(h_{n+1})]\varphi_1(h_{n+1})\}\|y_n\|^2 \\
& +2h_{n+1}\varphi_1(h_{n+1})\beta \max_{\hat{t}_n \leq t \leq t_{n+1}} \|y^h(t)\|^2 + 2(\gamma - \alpha\Gamma)h_{n+1},
\end{aligned} \tag{5.2}$$

which implies that

$$\|y_{n+1}\|^2 \leq \begin{cases} \sigma(h_{n+1}) \max_{\hat{n} \leq i \leq n} \|y_i\|^2 + r_1(h_{n+1}), & for\ \eta(\hat{t}_n) \geq 0, \\ \sigma(h_{n+1}) \max\left\{\max_{1 \leq i \leq n} \|y_i\|^2, \bar{\phi}^2\right\} + r_1(h_{n+1}), & for\ \eta(\hat{t}_n) < 0, \end{cases} \tag{5.3}$$

where

$$\sigma(h) = \frac{1 - \nu_1 h}{1 - \nu_2 h}, \quad r_1(h) = \frac{2h(\gamma - \alpha\Gamma)}{1 - 2h[\alpha(C_1 + \varphi_2(h) + 1]\varphi_1(h)},$$

$$\nu_1 = 2[\alpha(C_2 + \varphi_2(h)) - c_\pi^2\beta]\varphi_1(h), \quad \nu_2 = 2\alpha[C_1 + \varphi_2(h)]\varphi_1(h)$$

Note that $\hat{t}_n$ and $\hat{n}$ have been defined in Lemma 3.1. It is easy to verify that $0 < \sigma(h) < 1$ and $\sigma(h)$ is a strictly monotone decreasing function. The remaining part of this proof is analogous with that of Theorem 3.1 and we omit it here.□

As an application of the above theorem, we present the following example.

Example 5.1. Consider 2-stage algebraically stable Lobatto IIIC method

$$\begin{array}{c|cc} 0 & 1/2 & -1/2 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}, \tag{5.4}$$

which is analyzed as a example in Wen et al. [63]. First we have

$$\begin{aligned}
\|Y^{(n)}\|^2 &= \|y_{n+1} - h_{n+1}f(t_{n+1}, y_{n+1}, y^h(\cdot))\|^2 + \|y_{n+1}\|^2 \\
&= 2\|y_{n+1}\|^2 + h_{n+1}^2\|f(t_{n+1}, y_{n+1}, y^h(\cdot))\|^2
\end{aligned}$$

$$\begin{aligned}&-2h_{n+1}\Re e\langle y_{n+1}, f(t_{n+1}, y_{n+1}, y^h(\cdot))\rangle\\ &\geq 2\|y_{n+1}\|^2 - 2h_{n+1}\Re e\langle y_{n+1}, f(t_{n+1}, y_{n+1}, y^h(\cdot))\rangle.\end{aligned}$$

From condition (1.2), one gets

$$\Re e\langle y_{n+1}, f(t_{n+1}, y_{n+1}, y^h(\cdot))\rangle \leq \gamma + \alpha\|y_{n+1}\| + \beta \max_{\hat{t}_n \leq t \leq t_{n+1}} \|y^h(t)\|^2.$$

Combining the above two inequality to obtain

$$\begin{aligned}\Re e\langle Y^{(n)}, BY^{(n)}\rangle &= \frac{1}{2}\|Y^{(n)}\|^2\\ &\geq \|y_{n+1}\|^2 - h_{n+1}\left(\gamma + \alpha\|y_{n+1}\| + \beta \max_{\hat{t}_n \leq t \leq t_{n+1}} \|y^h(t)\|^2\right)\\ &= (1-\alpha h_{n+1})\|y_{n+1}\|^2 - h_{n+1}\gamma - h_{n+1}\beta \max_{\hat{t}_n \leq t \leq t_{n+1}} \|y^h(t)\|^2.\end{aligned}$$

Then (5.1) holds with

$$C_1 = 1, \quad C_2 = 0, \quad \varphi_1(h) = 1 - \alpha h, \quad \varphi_2(h) = 0, \quad \varphi_3(h) = \gamma h.$$

Consequently, for any variable stepsize, the 2-stage Lobatto IIIC method (5.4) with linear interpolation is dissipative under the same dissipative condition as the underlying system.

6. Applications to the Nicholson's blowflies models

The objective from now on is to show that the previous theory can be applied to several situations coming from applications. We report on numerical experiments where some concrete examples are considered.

6.1. Nicholson's blowflies equation with variable delay

We are first interested in considering a situation which takes into account the possible appearance of variable delay

$$y'(t) = -ay(t) + b(t)|y(\lambda t - \tau)|^k e^{-\xi(t)y(\lambda t-\tau)}, \qquad t \geq 0, \tag{6.1}$$

where $a > 0$, $0 < k \leq 1$, $0 < \lambda \leq 1$, $\tau \geq 0$, $b(t) > 0$, $\xi(t) \geq 0$ are continuous and $b(t) \leq \beta_1$, for any $t \geq 0$. The Nicholson's model of blowflies [16] is a particular case of (6.1) with $k = 1$ and $\lambda = 1$. Note that when $k = 0$, this equation reduces to

$$y'(t) = -ay(t) + b(t)e^{-\xi(t)y(\lambda t-\tau)}, \qquad t \geq 0, \tag{6.2}$$

which is Lasota and Wazewska model of production of blood cells [26] for $\lambda = 1$. These models with a constant delay have been studied in Caraballo et al. [6]. Both the results obtained in Caraballo et al. [6] and Proposition 1.1 here confirmed that the system (6.2) is dissipative for any $a > 0$. For the system (6.1), it can be obtained from [6] that the dissipativity condition is $\beta_1 < \sqrt{\frac{\lambda}{e}}a$ for $k = 1$ and $ek < a^2$ for $0 < k < 1$. However, from Proposition 1.1 we can verify that the dissipativity condition is

$$\begin{cases}\beta_1 < a, & \text{if } \ k = 1;\\ k < a, & \text{if } \ 0 \leq k < 1,\end{cases} \tag{6.3}$$

under which the system (6.1) is dissipative. To obtain this condition, we first need to note that the solution $y(t)$ of (6.1) satisfies $y(t) \geq 0$ for any $t \geq -\tau$. Then when $k = 1$ we have

$$\begin{aligned}\Re e\langle -ay(t) + b(t)|y(\lambda t-\tau)|e^{-\xi(t)y(\lambda t-\tau)}, y(t)\rangle &\leq -a|y(t)|^2 + \beta_1|y(\lambda t-\tau)||y(t)|\\ &\leq \left(-a + \frac{\beta_1}{2}\right)|y(t)|^2 + \frac{\beta_1}{2}|y(\lambda t-\tau)|^2,\end{aligned} \tag{6.4}$$

which implies that (1.2) holds with $\alpha = -a + \frac{\beta_1}{2}$ and $\beta = \frac{\beta_1}{2}$. Consequently, $\beta_1 < a$ implies the dissipativity criteria (1.5). If $0 \leq k < 1$, to obtain the dissipativity condition (6.3) we need the Young's inequality

$$xz \leq \frac{x^r}{r} + \frac{z^s}{s}, \qquad \text{for } x, z \geq 0, \ r, s > 1 \text{ with } r^{-1} + s^{-1} = 1. \tag{6.5}$$

Using this inequality to obtain

$$\begin{aligned}&\Re e\langle -ay(t) + b(t)|y(\lambda t-\tau)|^k e^{-\xi(t)y(\lambda t-\tau)}, y(t)\rangle\\ &\leq -a|y(t)|^2 + \beta_1|y(\lambda t-\tau)|^k|y(t)|\\ &\leq -a|y(t)|^2 + \left((1-k)\beta_1^{\frac{1}{1-k}} + k|y(\lambda t-\tau)|\right)|y(t)|,\end{aligned}$$

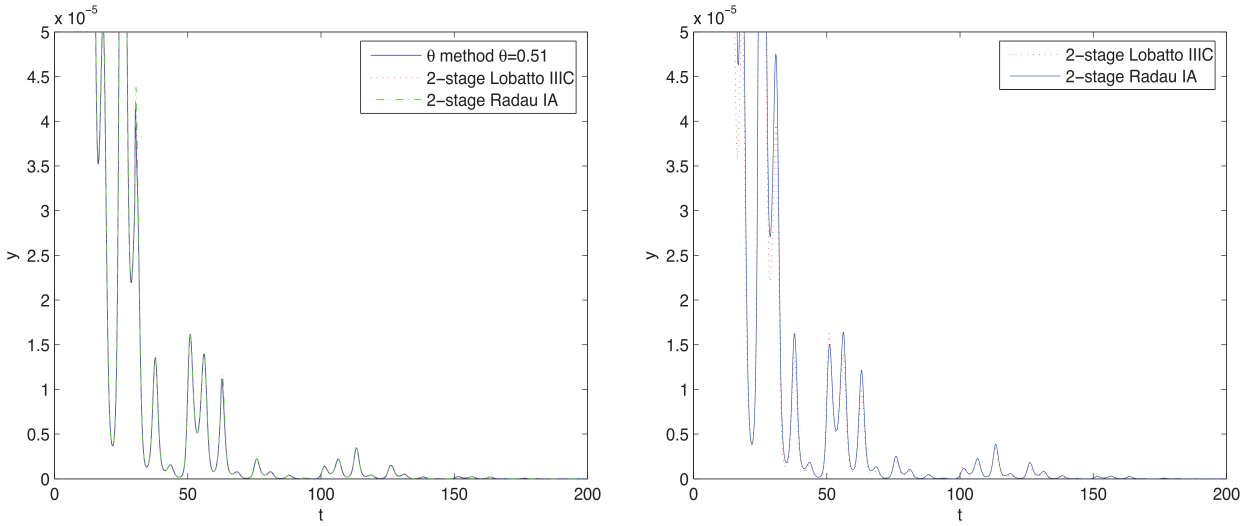

Fig. 6.1. Numerical solutions to (6.1), (6.7) with $k = 1$ computed by ODEs numerical methods with linear interpolation. Left: one-leg θ-method ($\theta = 0.51$), 2-stage Lobatto IIIC and 2-stage Radau IA with $h = 0.002$; Right: 2-stage Lobatto IIIC and 2-stage Radau IA with $h = 0.2$.

$$\leq \frac{(1-k)^2}{4\epsilon}\beta_1^{\frac{2}{1-k}} + \left(-a+\epsilon+\frac{k}{2}\right)|y(t)|^2 + \frac{k}{2}|y(\lambda t-\tau)|^2, \qquad \forall \epsilon > 0, \tag{6.6}$$

which implies that (1.2) holds with $\gamma = \frac{(1-k)^2}{4\epsilon}\beta_1^{\frac{2}{1-k}}$, $\alpha = -a+\epsilon+\frac{k}{2}$ and $\beta = \frac{k}{2}$. Then the condition $k < a$ implies that the dissipativity criteria (1.5) holds.

Now consider the numerical solution to equation (6.1) subject to the initial condition

$$y(t) = e^t \sin(t) + 1, \qquad t \leq 0. \tag{6.7}$$

The parameters a, k, and the functions $b(t)$, $\xi(t)$, are chosen such that the system is dissipative. We choose $a = 100$, $b(t) = 10e^{\cos(t)}$, $\xi(t) = t$, $\lambda = 1/2$, $\tau = 1$.

Example 6.1. Let us consider the case of $k = 1$. In this case, we have $\alpha = -100 + 5e$ and $\beta = 5e$ and therefore the system is dissipative. In fact, from Proposition 1.1 we further know that in view of $\gamma = 0$ the system is asymptotic to a fixed point $y = 0$.

We now apply three ODEs numerical methods, the one-leg θ-method ($\theta = 0.51$), the 2-stage Lobatto IIIC method and the 2-stage Radau IA method, with linear interpolation to problem (6.1) and (6.7).

Using $\theta = 0.51$, it is easy to verify $\alpha + \frac{1}{(2\theta-1)^2}\beta > 0$. Then the result obtained in Wen et al. [63] can not be applied to this case. For it our result will come in handy. It follows from Example 3.1 that one-leg θ-method with $\theta = 0.51$ will preserve the dissipativity of the underlying system (6.1) if the stepsizes satisfy the following constraint condition:

$$0.0004707765 \leq h_n < 0.0026727252, \qquad \forall n \geq 1. \tag{6.8}$$

Therefore, we first apply the one-leg θ-method ($\theta = 0.51$) with linear interpolation to solve the problem (6.1), where the stepsize $h = 0.002$. The numerical results are displayed in Fig 6.1.

Since the 2-stage Lobatto IIIC method with linear interpolation can preserve the dissipativity of the underlying system for any stepsize, this method with stepsize $h = 0.002$ and $h = 0.2$ is also used to solve the problem (6.1), (6.7). The numerical results are also displayed in Fig. 6.1.

For the 2-stage Radau IA method, it is not hard to known from Example 4.1 that this method with linear interpolation can preserve the dissipativity of the underlying system, since $\alpha + 4\beta = -100 + 25e < 0$. Notice that we can not reach any conclusion for this case from the previous results in the literatures. The numerical results confirm our theoretical analysis.

6.2. Nicholson's blowflies equation with distributed delay

The next example is about the integro-differential Nicholson's model (see the review paper [4])

$$y'(t) = -ay(t) + b(t)\int_{-\tau(t)}^{0} |y(t+s)|^k e^{-\xi(t+s)y(t+s)} ds, \qquad t \geq 0, \tag{6.9}$$

where $a > 0$, $0 < k \leq 1$, $\tau(t) \geq 0$, $b(t) > 0$, $\xi(t) \geq 0$ are continuous and $\tau(t) \leq \tau$, $b(t) \leq \beta_1$, for any $t \geq 0$. This model with a constant delay has been studied in Caraballo et al. [6]. The arguments in the above subsection can be used to derive the dissipativity condition for this model. The dissipativity condition is the same as (6.3) except β_1 being replaced by $\beta_1\tau$.

Now we consider the numerical solution. We still apply the one-leg θ-methods, 2-stage Lobatto IIIC method and 2-stage Radau IA method to problem (6.9) subject to the initial condition (6.7).

When the one-leg θ-method with $\theta = 0.51$ is applied to solve (6.9), we have

$$y_{n+1} = y_n + h\left[-ay_n^\theta + b(t_n^\theta)\int_{-\tau(t_n^\theta)}^0 |y(t_n^\theta + s)|^k e^{-\xi(t_n^\theta+s)y(t_n^\theta+s)}ds\right] \tag{6.10}$$

where $y_n^\theta = \theta y_{n+1} + (1-\theta)y_n$ and $t_n^\theta = \theta t_{n+1} + (1-\theta)t_n$. Let $m_n = \lfloor (t_n^\theta - \tau(t_n^\theta))/h \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less or equal to x. Then if the linear interpolation is used in this method, the integral on the right hand side of the equation (6.10) can be approximated by the following formula, where $g(t,y) = |y|^k e^{-\xi(t)y}$ and $\zeta(t) = t - \tau(t)$,

$$\begin{aligned}\int_{-\tau(t_n^\theta)}^0 g(t_n^\theta + s, y(t_n^\theta + s))ds \approx{}& \frac{1}{2}g\Big(\zeta(t_n^\theta), (\zeta(t_n^\theta) - t_{m_n})y_{m_n+1} + (t_{m_n+1} - \zeta(t_n^\theta))y_{m_n}\Big)(t_{m_n+1} - \zeta(t_n^\theta))\\ &+\frac{1}{2}\big[h + (t_{m_n+1} - \zeta(t_n^\theta))\big]g(t_{m_n+1}, y_{m_n+1}) + h\sum_{i=m_n+2}^{n-1} g(t_i, y_i)\\ &+\frac{1}{2}(1+\theta)hg(t_n, y_n) + \frac{1}{2}\theta hg(t_n^\theta, y_n^\theta).\end{aligned} \tag{6.11}$$

For solving the obtained nonlinear algebraic equations, we use Newton iterative method.

Now applying 2-stage Lobatto IIIC method with the linear interpolation and 2-stage Radau IA method with the linear interpolation to (6.9) yields the corresponding nonlinear algebraic systems. For solving the nonlinear equations, we consider the following iteration scheme where on iteration $\mathcal{N}$ we have

$$\begin{aligned} Y_i^{(n,\mathcal{N})} &= y_n + h_{n+1}\sum_{j=1}^s a_{ij}\big[-aY_j^{(n,\mathcal{N})} + b(t_{n,j})G_j^{(n,\mathcal{N}-1)}\big], \quad i = 1,\dots,s,\\ y_{n+1}^{(\mathcal{N})} &= y_n + h_{n+1}\sum_{j=1}^s b_j\big[-aY_j^{(n,\mathcal{N})} + b(t_{n,j})G_j^{(n,\mathcal{N}-1)}\big],\end{aligned} \tag{6.12}$$

with the initial guess $y_{n+1}^{(0)} = y_n$. Here, $G_j^{(n,\mathcal{N}-1)}$ is defined by the following

$$\begin{aligned} G_j^{(n,\mathcal{N}-1)} ={}& \frac{1}{2}g\Big(\zeta(t_{n,j}), (\zeta(t_{n,j}) - t_{m_n^j})y_{m_n^j+1} + (t_{m_n^j+1} - \zeta(t_{n,j}))y_{m_n^j}\Big)(t_{m_n^j+1} - \zeta(t_{n,j}))\\ &+\frac{1}{2}\Big[h + (t_{m_n^j+1} - \zeta(t_{n,j}))\Big]g\Big(t_{m_n^j+1}, y_{m_n^j+1}\Big) + h\sum_{i=m_n^j+2}^{n-1} g(t_i, y_i)\\ &+\frac{1}{2}(1 + c_j(1-c_j))hg(t_n, y_n) + \frac{1}{2}c_j^2 hg(t_{n+1}, y_{n+1}^{(\mathcal{N}-1)}).\end{aligned} \tag{6.13}$$

where $m_n^j = \lfloor (t_{n,j} - \tau(t_{n,j}))/h \rfloor$. Following the approach designed by Enright and Hu [10] for continuous RKMs, we can easily prove that the iteration (6.12) is convergent for sufficiently small h (see also [54]).

Example 6.2. Let $a = 100$, $b(t) = 10e^{\cos(t)}$, $\tau(t) = \frac{1}{3}(\sin(t) + 2)$, $\xi(t) = |t|$. In this case, we have $\alpha = -100 + 5e$ and $\beta = 5e$ when $k = 1$. Following the approach used in Subsection 6.1, we know that the θ-method ($\theta = 0.51$) with linear interpolation will preserve the dissipativity of the underlying system (6.9) if the stepsizes satisfy the constraint condition (6.8), the 2-stage Lobatto IIIC method with linear interpolation and the 2-stage Radau IA method with linear interpolation can preserve the dissipativity of the underlying system for any stepsize. The numerical solutions to the equation (6.9) with the initial condition (6.7) obtained by the three numerical methods with different stepsizes are displayed in Fig. 6.2. Observe that when $h = 0.002$, the three numerical methods can really simulate the behavior of the solution to (6.9), (6.7). However, when $h = 0.1$, the θ-method ($\theta = 0.51$) with linear interpolation can be seen to give temporal oscillations and negative values near the points $t = 0$. In fact, this motivate us to consider the positivity properties of numerical methods which will be our future work.

Example 6.3. Now consider the case $0 < k < 1$. Let $k = 0.5$, $a = 100$, $b(t) = 10e^{\cos(t)}$, $\tau(t) = \frac{1}{3}(\sin(t) + 2)$, $\xi(t) = |t|$. Then $\alpha = -100 + \epsilon + 0.25$ for any $\epsilon > 0$ and $\beta = 0.25$. The 2-stage Lobatto IIIC method with linear interpolation and the 2-stage Radau IA method with linear interpolation are dissipative for any stepsize. For the θ-method with $\theta = 0.51$, the dissipativity is guaranteed if the stepsize is restricted so that

$$0.00040116 \le h_n < 0.19038553, \qquad \forall n \ge 1. \tag{6.14}$$

holds. The numerical results are presented in Fig. 6.3.

To illustrate the convergence of these methods, in this example, we also compute the errors. We let the numerical solutions obtained by these numerical methods with $h = 0.002$ be the reference solutions. The errors at $t = 100$ and the orders of convergence are presented in Table 6.1. Observe that the θ-method ($\theta = 0.51$) with linear interpolation has convergence of order 1 and the other two methods with linear interpolation have convergence of order 2. A point which is especially

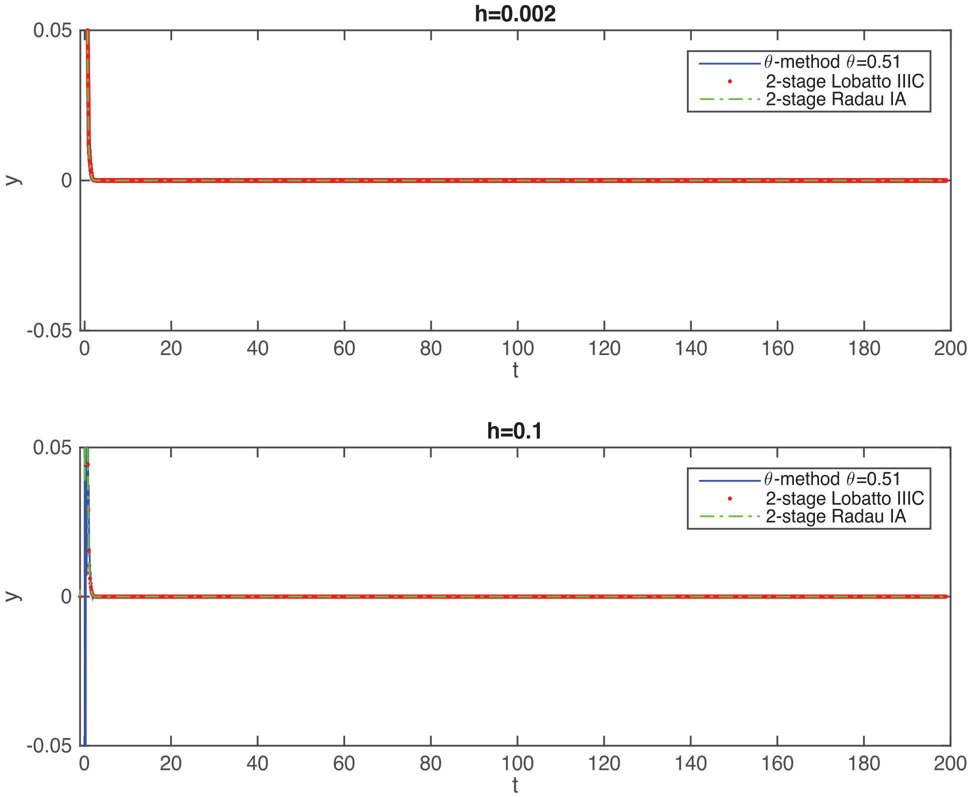

Fig. 6.2. Numerical solutions to (6.9), (6.7) with $k = 1$ computed by one-leg θ-method ($\theta = 0.51$), 2-stage Lobatto IIIC and 2-stage Radau IA with stepsizes $h = 0.002$ and $h = 0.1$.

Table 6.1
The numerical solutions and the errors produced by the θ-method ($\theta = 0.51$), the 2-stage Lobatto IIIC method and the 2-stage Radau IA method with linear interpolation applied to (6.9), (6.7) with $k = 0.5$, and the orders of these methods, where the numerical solutions obtained by these numerical methods with $h = 0.002$ are the reference solutions.

Methods	h	Numerical solutions	Errors	Orders
	0.1	5.360983×10^{-3}	6.05434×10^{-4}	–
θ-method	0.05	5.061867×10^{-3}	3.06318×10^{-4}	0.982940
($\theta = 0.51$)	0.025	4.904889×10^{-3}	1.49340×10^{-4}	1.03643
	0.1	4.740461×10^{-3}	1.869×10^{-6}	–
2-stage	0.05	4.742121×10^{-3}	0.209×10^{-6}	3.160692
Lobatto IIIC	0.025	4.742365×10^{-3}	0.035×10^{-6}	2.578076
	0.1	4.734412×10^{-3}	7.914×10^{-6}	–
2-stage	0.05	4.740661×10^{-3}	1.665×10^{-6}	2.248885
Radau IA	0.025	4.741973×10^{-3}	0.353×10^{-6}	2.237782

to be noted in connection with these numerical results is that the 2-stage Lobatto IIIC method with linear interpolation exhibits higher accuracy and higher convergence order than the 2-stage Radau IA method with linear interpolation, although the 2-stage Lobatto IIIC method has convergence of order 2 and the 2-stage Radau IA method has convergence of order 3 when they are applied to ODEs. We think that this arises mainly because the 2-stage Lobatto IIIC method with linear interpolation has better long-time stability property than the 2-stage Radau IA method with linear interpolation.

6.3. Diffusive Nicholson's blowflies equation with multiple delays

In this subsection we will consider the diffusive Nicholson's blowflies equation with multiple delays

$$\frac{\partial u(x,t)}{\partial t} = \nu \frac{\partial^2 u(x,t)}{\partial x^2} - au(x,t) + \sum_{i=1}^{k} b_i(t) u(x, t-\tau_i) e^{-\xi u(x,t-\tau_i)},$$
$$t \geq 0, \quad x \in (0,1), \tag{6.15}$$

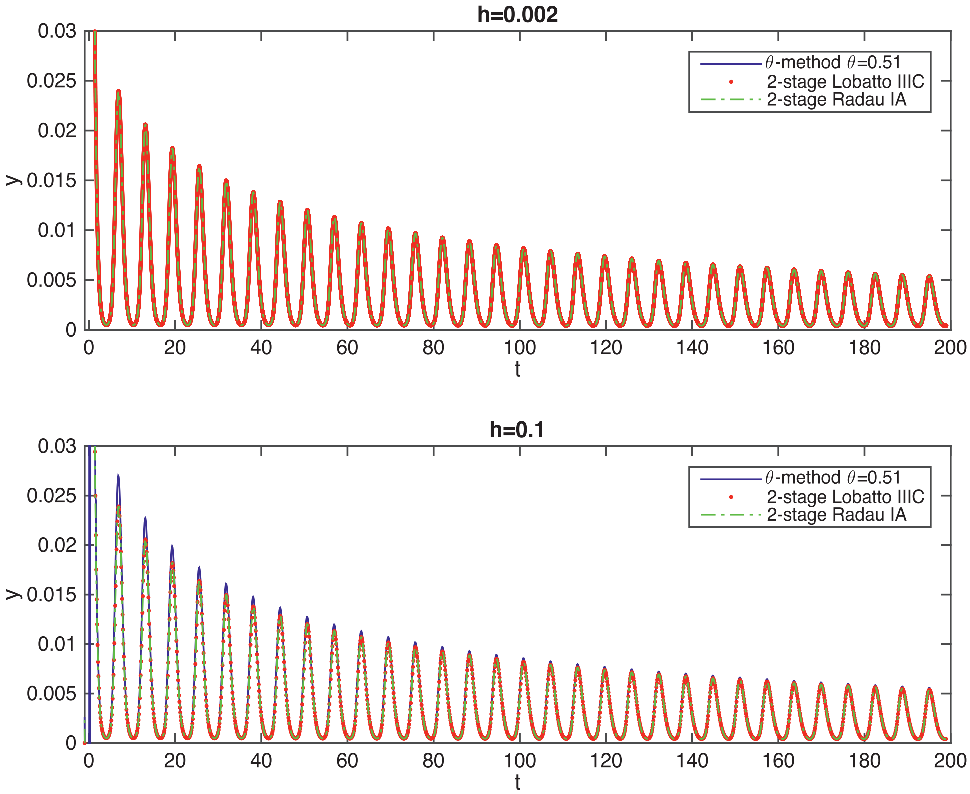

Fig. 6.3. Numerical solutions to (6.9), (6.7) with $k = 0.5$ computed by one-leg θ-method ($\theta = 0.51$), 2-stage Lobatto IIIC and 2-stage Radau IA with stepsizes $h = 0.002$ and $h = 0.1$.

subject to

$$u(0,t) = u(1,t) = 0, \qquad t \geq 0, \tag{6.16}$$

$$u(x,t) = \phi(x,t), \qquad t \in [-\tau, 0], \quad x \in [0,1], \tag{6.17}$$

where $\nu > 0$, $a > 0$, $\tau_i \geq 0$ $(i = 1, \ldots, k)$, $\xi \geq 0$, $\tau = \max_{1\leq i\leq k}\{\tau_i\}$, $b_i(t) > 0$ $(i = 1, \ldots, k)$ are continuous and $b_i(t) \leq \beta_i$, for any $t \geq 0$. This equation has been widely discussed in recent years (see, for example, [32,33,39,42,43]). Using the boundary condition (6.16), we can verify the condition (1.2) with $\gamma = 0$, $\alpha = -\nu\pi^2 - a + \sum_{i=1}^k \beta_i/2$ and $\beta = \sum_{i=1}^k \beta_i/2$. Then in view of Proposition 1.1, the following condition implies that the system (6.15) and (6.16) is dissipative

$$-\nu\pi^2 - a + \sum_{i=1}^k \beta_i < 0. \tag{6.18}$$

We observe that $\gamma = 0$ which implies that the solution is asymptotically stable, i.e. $\lim_{t\to\infty} \|u(x,t)\|_{L^2} = 0$, under the condition (6.18). After application of the numerical method of lines, we obtain the following delay differential equations of the form

$$U_i'(t) = \Delta x^{-2}[U_{i-1}(t) - 2U_i(t) + U_{i+1}(t)] - aU_i(t) + \sum_{j=1}^k b_j(t)U_i(t-\tau_j)e^{-\xi U_i(t-\tau_j)}, \qquad t \geq 0, \tag{6.19}$$

$$U_0(t) = U_M(t) = 0, \quad t \geq 0, \tag{6.20}$$

$$U_i(t) = \phi(x_i, t), \quad i = 0, 1, \ldots, M, \quad t \in [-\tau, 0] \tag{6.21}$$

where $M = 1/\Delta x$, $x_i = i\Delta x$ and $U_i(t)$ is meant to approximate the solution of (6.15) at the point (t, x_i). We take $M = 800$ for the numerical method of lines.

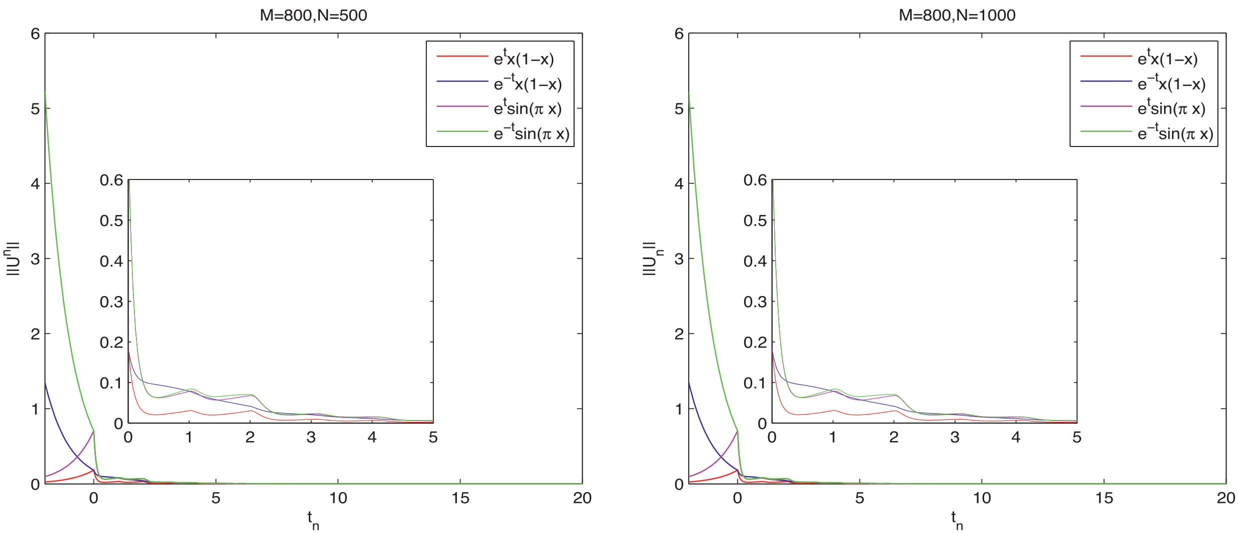

Fig. 6.4. Numerical results of the θ-method ($\theta = 0.51$) connecting the method of lines with the stepsizes $h = 0.04$ and $h = 0.02$ when applied to problem (6.15)–(6.17) subject to four different initial conditions (6.22)–(6.25), where $Nh = 20$. Left: $h = 0.04$; Right: $h = 0.02$.

The purpose of this numerical example is to affirm that (2.10) holds for any bounded initial functions, that is, the system possesses a bounded absorbing set. For this purpose, we give the following four initial functions

$$\phi(x,t) = e^t x(1-x), \qquad t \in [-\tau, 0], \quad x \in [0,1]; \tag{6.22}$$

$$\phi(x,t) = e^{-t} x(1-x), \qquad t \in [-\tau, 0], \quad x \in [0,1]; \tag{6.23}$$

$$\phi(x,t) = e^t \sin(\pi x), \qquad t \in [-\tau, 0], \quad x \in [0,1]; \tag{6.24}$$

$$\phi(x,t) = e^{-t} \sin(\pi x), \qquad t \in [-\tau, 0], \quad x \in [0,1]. \tag{6.25}$$

From the previous two experiments, observe that the numerical solutions produced by the θ-method ($\theta = 0.51$) are dissipative and convergent when the stepsizes satisfy the condition (3.26) given in this paper as well as 2-stage Lobatto IIIC method and 2-stage Radau IA method. So, for simplicity, in following examples we consider only using the θ-method with $\theta = 0.51$ to compute the approximation solutions.

Example 6.4. Let $\nu = 1$, $a = 0.5$, $\xi = 1$, $k = 2$, $b_1(t) = 0.5$, $b_2(t) = 1.5$, $\tau_1 = 1$, and $\tau_2 = 2$. Then the condition (6.18) holds and hence the solution is asymptotically stable, i.e. $\lim_{t\to\infty} \|u(x,t)\|_{L^2} = 0$. Now consider the discrete norm

$$\|U^n\|^2 = \Delta x \sum_{i=1}^{M-1} \left(U_i^n\right)^2, \qquad n \geq -2/h, \tag{6.26}$$

where v_i^n denotes the numerical solution which is produced by θ-method ($\theta = 0.51$) approximating $u(x_i, t_n)$ with $t_n = nh$. According to our analysis, we have came to the conclusion that

$$\lim_{n\to\infty} \|U^n\| = 0$$

if the stepsize h satisfies $0.004276 < h < 0.04015$. The numerical results of the θ-method ($\theta = 0.51$) connecting the method of lines with the stepsizes $h = 0.04$ and $h = 0.02$ when applied to problem (6.15)–(6.17) subject to four different initial conditions (6.22)–(6.25) are shown in Fig. 6.4.

Example 6.5. It should be pointed out that if the boundary condition (6.16) is changed into the following condition

$$u(0,t) = u(1,t) = u^*, \qquad t \geq 0, \tag{6.27}$$

where u^* is a constant, we need to use the change of variables $\nu = u - u^*$. The function ν will satisfy

$$\frac{\partial \nu(x,t)}{\partial t} = \nu \frac{\partial^2 \nu(x,t)}{\partial x^2} - a(\nu(x,t) + u^*) + \sum_{i=1}^{k} b_i(t)(u^* + \nu(x, t-\tau_i))e^{-\xi(u^*+\nu(x,t-\tau_i))},$$
$$t \geq 0, \quad x \in (0,1), \tag{6.28}$$

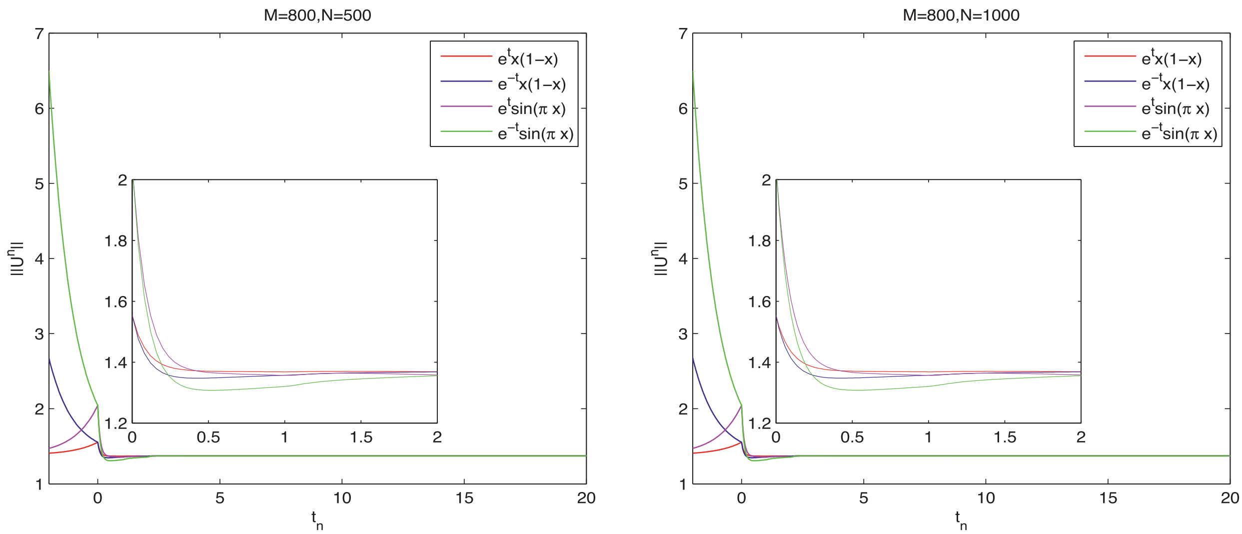

Fig. 6.5. Numerical results of the θ-method ($\theta = 0.51$) connecting the method of lines with the stepsizes $h = 0.04$ and $h = 0.02$ when applied to problem (6.15) subject to the boundary condition (6.27) and the initial condition $u(x,t) = \phi(x,t) + u^*$, $x \in [0,1], t \in [-2,0]$, where $Nh = 20$. Left: $h = 0.04$; Right: $h = 0.02$.

and the boundary condition (6.16). Then the condition (1.2) can be verified with $\gamma = (\frac{a}{4\epsilon_1} + \frac{\sum_{i=1}^k \beta_i}{4\epsilon_2})u^{*2}$, $\alpha = -\nu\pi^2 - a + \epsilon_1 + \epsilon_2 + \sum_{i=1}^k \beta_i/2$ and $\beta = \sum_{i=1}^k \beta_i/2$ for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$. Let the involved parameters be the same values as in **Example 6.5**. We still use the θ-method ($\theta = 0.51$) connecting the method of lines to solve the problem (6.28) with the boundary condition (6.16), and the initial condition (6.17) where ϕ is given by (6.22) or (6.23) or (6.24) or (6.25). The numerical solutions U_i^n approximating to the solutions of the original problem (6.15) subject to the boundary condition (6.27) and the initial condition $u(x,t) = \phi(x,t) + u^*$, $x \in [0,1], t \in [-2,0]$ are shown in Fig. 6.5, where $u^* = 2\ln 2$. Observe that $\lim_{n\to\infty} \|U^n\| = u^*$.

7. Concluding remarks

In this paper, we have presented some new dissipative results for the variable-stepsize RK methods applied to VFDEs. It well known that an efficient numerical method should be of variable-stepsize. To the best of our knowledge, however, the existing numerical dissipative results for VFDEs are all for the case of fixed-stepsize. Accordingly, the derived methods in this paper and their dissipative results gave a large improvement to the existing results.

To weaken the condition $\alpha + p\beta < 0$, we considered two approaches. One is based on Theorem 4.1 by which we can compute the values of C_1 and C_2 more accurately, and therefor p. The other is based on the condition (5.1) which has been considered in Li [29]. By use of this result, we prove that the 2-stage Lobbatto IIIC method with piecewise linear interpolation has $p = 1$, that is, it is dissipative for any variable stepsize under the same dissipative condition as the underlying system.

When $\alpha + p\beta > 0$, no previous results can be applied. In this paper, for this case, we considered the conditional dissipativity which implies the method can preserve exactly the dissipativity of the underlying system under a stepsize restriction and showed the conditional dissipativity of some $(k,l)-$algebraically stable RK methods.

Our findings extend and improve earlier results reported in Wen et al. [63]. Specializing our results to the DDEs $y'(t) = f(t, y(t), y(t-\tau))$, we find our results to be slightly weaker than those in Huang [20], Wen et al. [61]. As a conclusion, we would like to point out the precise differences.

(1) For DDEs with a constant delay, the special case of VFDEs (1.1), the author in Huang [20] studied the dissipativity of (k,l)-algebraically stable RK method with linear interpolation in a Hilbert space and obtained the sufficient conditions: $k < 1$ and $(\alpha + \beta)h < l$. Especially, a sufficient condition for a consistent, *DJ*-irreducible, algebraically stable RK method with linear interpolation which can preserve the dissipatvity of the underlying system is $|R(\infty)| < 1$.

(2) For DIDEs with a constant delay, Gan [12] studied the dissipativity of θ-methods. Wen et al. [61] studied the dissipativity of RK methods for NDIDEs with a constant delay in a finite-dimensional space. As a corollary of their results, it is revealed that a consistent, $b_j > 0$, $j = 1, 2, \ldots, s$, algebraically stable RK method with linear interpolation can preserve the dissipatvity of the underlying system.
These strong property could be derived thanks to the relatively simple structure of the problem, a constant delay problem. The approach cannot be generalized to more complex problems, for example, a general variable delay problem.

Table 7.1
Dissipativity results of RKMs for VFDEs satisfying $\alpha + \beta < 0$ obtained in the present paper with comparison to those obtained in Wen et al. [63], where $q = l + \frac{1-k}{2(C_1 - C_2)}$.

Methods	Assumptions	Present results (variable stepsize h_n)	Results in Wen et al. [63] (constant stepsize h)
(k, l)-algebraically stable RKMs satisfying (3.1), $p = \frac{dc_\pi^2}{C_1 - C_2}$	$\alpha + p\beta < 0$	$\alpha h_n \le l,\ (\alpha + p\beta)h_n < q$	$(\alpha + p\beta)h < 2l$
	$\alpha + p\beta \ge 0$	$\frac{l}{\alpha} \le h_n < \frac{q}{\alpha + p\beta}$	no results
(k, l)-algebraically stable RKMs satisfying (3.1), $\delta = \lvert R(\infty)\rvert < 1,\ p = \frac{dc_\pi^2}{C_1 - C_2}$		$C_1 = \frac{1-\delta}{\sum_{j=1}^{s} d_j^{-1} u_j^2}$, $C_2 = \delta C_1$, $[u_1, \dots, u_s] = \lim_{\epsilon \to 0} b^T (A + \epsilon I)^{-1}$	$C_1 = \frac{(1-\delta)\left(\min_{1 \le j \le s} b_j\right)}{\lVert b^T A^{-1} \rVert_0^2}$, $C_2 = \delta C_1$
Algebraically stable RKMs satisfying (3.1), $p = \frac{c_\pi^2}{C_1 - C_2}$	$\alpha + p\beta < 0$	$\forall h_n > 0$	$\forall h > 0$
	$\alpha + p\beta \ge 0$	$\frac{l}{\alpha} \le h_n < \frac{q}{\alpha + p\beta}$	no results
Algebraically stable RKMs satisfying (5.1), $p = \frac{c_\pi^2}{C_1 - C_2}$	$\alpha + p\beta < 0$	$\forall h_n > 0$	no results

(3) For DDE with a proportional delay, Gan [13] studied the dissipativity of implicit Euler method by transforming this equation into a constant delay differential equation. So far we have not seen in literature other numerical dissipativity results for nonlinear proportional delay differential equations.

(4) In [63], the authors also investigated the dissipativity of RK methods for the general VFDEs in a Hilbert space. To compare our results with their results clearly, we summarize the dissipativity results of RK methods for VFDEs obtained in this paper and in Wen et al. [63] and list them in Table 7.1.

We note that the convergence of RK methods (2.1) for VFDEs has been reported in Li [27] and Li and Li [31]. From [27] and [31], we know that under some conditions, the errors of B-consistent RK methods (2.1) of order q with piecewise Lagrangian interpolation of degree not less than $q-1$ can be estimated by

$$\lVert y_n - y(t_n) \rVert \le C_0(t_n) \max_{-\tau \le t \le 0} \lVert \psi(t) - \phi(t) \rVert + C(t_n) \left(\max_{0 \le i \le n-1} h_i \right)^q, \tag{7.1}$$

where ψ is an initial approximation to $\phi(t)$. We also note that the error bounds of general linear methods (GLMs) for VFDEs have been derived in Li [28]. The dissipativity of GLMs for VFDEs is still open, although the dissipativity of one-leg methods, linear multistep methods and multistep Runge-Kutta methods, which are special cases of GLMs, for special FDEs has been investigated (see, e.g., [21,34,37,46]). Studying the dissipativity of GLMs for VFDEs will be our future work.

It should be pointed out that another class of extending RK methods, the functional continuous RK (FCRK) methods, was introduced in Maset et al. [35] (see, also, the survey [3]). Recently, the global errors of FCRK methods are analyzed in Maset and Zennaro [36] and have similar behavior with (7.1). However, the dissipativity of FCRK for VFDEs (1.1) is still open and will be a focus of our future reserch.

In this paper, we have applied our theoretical results to nonlinear Nicholson blowflies, including variable delay model, integro-differential Nicholson's model, and diffusive model with multiple delays, and implemented various numerical experiments for RK methods for these models. For all these models these experiments exactly verify the theoretical results. It is noteworthy that the results obtained in this paper can be easily applied to other VFDEs mathematical models in science and physics.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Wansheng Wang: Conceptualization, Methodology, Software, Writing - original draft, Visualization, Investigation. **Chengjian Zhang:** Methodology, Writing - review & editing, Investigation.

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